On the Stability of Fuzzy Systems

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***Abstract*— This paper studies the global asymptotic stability of a class of fuzzy systems. It demonstrates the equivalence of stability properties of fuzzy systems and linear time invariant (LTI) switching systems. A necessary condition and a sufficient condition for the stability of such systems are given, and it is shown that under the sufficient condition, a common Lyapunov function exists for the LTI subsystems. A particular case when the system matrices can be simultaneously transformed to normal matrices is shown to correspond to the existence of a common quadratic Lyapunov function. A constructive procedure to check the possibility of simultaneous transformation to normal matrices is provided.**

***Index Terms*—Asymptotic stability, switching systems.**

I. Introduction

RECENTLY, fuzzy control is being used in many practical industrial applications. One of the first questions to be answered in this context is the stability of the fuzzy system. In recent literature, Tanaka and Sugeno [3], have provided a sufficient condition for the asymptotic stability of a fuzzy system in the sense of Lyapunov through the existence of a common Lyapunov function for all the subsystems. Tanaka and Sano [4] have extended this to robust stability in case of systems with premise-parameter uncertainty. The model of the fuzzy system, considered in these papers, is that proposed by Takagi and Sugeno [2], which can be shown to be equivalent in stability to a *switching* system with linear time invariant (LTI) subsystems. These switching systems turn out to be a particular class among linear time varying (LTV) systems. The classical theory of LTV systems is discussed in [8, Sec. 9], and recent advances in LTV systems are in [9]. Narendra and Balakrishnan [6] have provided a simple sufficient condition for the stability of the switching system.

This paper discusses some necessary and some sufficient conditions for global asymptotic stability of a fuzzy system. Section II defines the fuzzy system model and the model of the corresponding *switching* system and global asymptotic stability of these systems. Section III shows the equivalence of these two systems regarding stability, and further sections consider switching systems only. Section IV discusses the main results of this paper. A sufficient condition is provided for the asymptotic stability of the system. It is shown that when this condition is satisfied, there exists a common Lyapunov function for the subsystems. Section V deals with a special case which leads to simultaneous normalization of the system matrices. In this case, the subsystems have a common *quadratic* Lyapunov function. Also, a constructional procedure is provided to check whether the theorem's premise is satisfied. An example illustrates the procedure. This result can be viewed as a generalization of the result in [6], a sufficient condition for stability of switching systems.

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II. Notations and Definitions

This section outlines the mathematical model of a free fuzzy system and that of the corresponding *switching* system. Stability of these systems in the asymptotic sense is also defined.

The Takagi and Sugeno [2] model for the fuzzy system is chosen. Let the system state vector at time instant k be $\overline{x}(k) = [x_1(k) \dots x_n(k)]^T$ where $x_1(k) \cdots x_n(k)$ are the state variables of the system at time instant k. Then the free fuzzy system is defined by the implications below

$$\begin{aligned} R^i\colon\ &\text{IF}\,(x_1(k)\,\text{is}\,\mathcal{S}_1^i,\text{AND}\ \cdots\ \text{AND}\,x_n(k)\,\text{is}\,\mathcal{S}_n^i)\\ &\text{THEN}\,\overline{x}(k+1) = A_i\overline{x}(k) \end{aligned} \tag{1}$$

for $i = 1 \cdots N$. Here, $\mathcal{S}_j^l$ is the fuzzy set corresponding to the state variable x_j and implication R^l. $A_i \in \mathcal{R}^{n\times n}, i = 1 \cdots N$ are the system characteristic matrices. The *truth value* of the implication R^i at time instant k denoted by $w_i(k)$ is defined as

$$w_i(k) = \wedge(\mu_{\mathcal{S}_1^i}(x_1(k)), \cdots, \mu_{\mathcal{S}_n^i}(x_n(k))$$

where $\mu_{\mathcal{S}}(x)$ is the membership function value of the fuzzy set $\mathcal{S}$ at the position x and $\wedge$ is an operator satisfying

$$\min(l_1, \cdots, l_n) \geq \wedge(l_1, \cdots, l_n) \geq 0.$$

Usually $\wedge$ is taken to be the minimum operator which gives the minimum of its operands. Then, at instant k the state vector is updated according to

$$\begin{aligned} \overline{x}(k+1) &= \frac{\left(\sum_{i=1}^{N} w_i(k)A_i\overline{x}(k)\right)}{\sum_{i=1}^{N} w_i(k)}\\ &= \sum_{i=1}^{N} \alpha_i(k)A_i\overline{x}(k); \quad \alpha_i(k) = \frac{w_i(k)}{\sum_{i=1}^{N} w_i(k)}. \end{aligned} \tag{2}$$

A fuzzy system is completely represented by the set of characteristic matrices $\mathcal{A} = [A_1, \cdots, A_n]$ and the fuzzy sets $\mathcal{S}_j^l, l = 1, \cdots, N; j = 1, \cdots, n$. Corresponding to this free fuzzy system, a corresponding *switching* system is described below.

The state update at time instant k is given as

$$\overline{x}(k+1) = A\overline{x}(k) \tag{3}$$

where $A \in \mathcal{A}$ (i.e., it is one of the matrices $A_1, A_2, \cdots, A_n$). Also, define $\forall k \geq 1$

$$\mathcal{A}_k = \underbrace{\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}}_{k\,\text{times}}$$

where the Cartesian product is defined to be the multiplication of matrices in the same order. The following are definitions of global asymptotic stability of these systems.

Definition 1) The fuzzy system described in (2) is globally asymptotically stable if

$$\overline{x}(k) \to 0 \quad \text{as} \quad k \to \infty \tag{4}$$

or, equivalently, there exists $||\cdot||$, a norm on $\mathcal{R}^n$

$$||\overline{x}(k)|| \to 0 \quad \text{as} \quad k \to \infty$$

for all initial values $x(0) \in \mathcal{R}^n$ and for all possible fuzzy sets $\mathcal{S}_i^j, \forall i = 1 \cdots N, \forall j = 1 \cdots n$.

Definition 2) The switching system described in (3) is globally asymptotically stable if

$$\begin{aligned} &\overline{x}(k+1) = A(k)\overline{x}(0) \to 0 \\ &\text{as} \quad k \to \infty; \quad \forall \overline{x}(0) \in \mathcal{R}^n \end{aligned} \tag{5}$$

where $A(k) \in \mathcal{A}_k$. Equivalently

$$A(k) \to 0 \quad \text{as} \quad k \to \infty; \quad A(k) \in \mathcal{A}_k. \tag{6}$$

For any matrix $A \in \mathcal{R}^n$, let $\rho(A)$ be the spectral radius of A (i.e., the largest magnitude of the eigen values of A). Let $\rho(\mathcal{A}_k)$ be defined as

$$\rho(\mathcal{A}_k) = \max\{\rho(A) : A \in \mathcal{A}_k\}.$$

Let $||\cdot||$ be a matrix norm on $\mathcal{R}^n$ and $\sigma(A)$ be the largest singular value of A. Then,

$$\begin{aligned} \sigma(A) &= \sqrt{\rho(A^T A)} \\ &= ||A||_{\text{sp}} \end{aligned} \tag{7}$$

where $||\cdot||_{\text{sp}}$ is the spectral norm on a matrix. Also, let $\sigma(\mathcal{A}_k) = \max\{\sigma(A) : A \in \mathcal{A}_k\}$. Let $(A)_{ij}$ represent the element a_{ij} where $A = [a_{ij}]$. Let A^* denote the conjugate transpose of A.

III. Equivalence of the Stability of Switching and Fuzzy Systems

This section illustrates the equivalence of the stability of a fuzzy system and its corresponding switching system. A necessary condition for stability of either of these systems is also given. The following theorem illustrates some equivalent statements about the stability of the switching system. The proof might be hidden in some textbook; we give it here for completeness.

Theorem 3.1: The following are equivalent:

1) the switching system in (3) is globally asymptotically stable as in Definition 2;
2) $\sigma(\mathcal{A}_k) \to 0$ as $k \to \infty$;
3) $\rho(\mathcal{A}_k) \to 0$ as $k \to \infty$.

Proof: (1) ⇒ (2)

By (6), $A(k) \to 0$ as $k \to \infty; A(k) \in \mathcal{A}_k$. By (7), $\sigma(\cdot)$ is a matrix norm and, hence

$$\sigma(A(k)) \to 0 \quad \text{as} \quad k \to \infty; \quad \forall A(k) \in \mathcal{A}_k.$$

In particular, $\max_{A(k) \in \mathcal{A}_k} [\sigma(A(k))] \to 0$ as $k \to \infty$. Hence

$$\sigma(\mathcal{A}_k) \to 0; \quad \text{as} \quad k \to \infty.$$

■

(2) → (3)

Now

$$\begin{aligned} \sigma(\mathcal{A}_k) &= \max\{\sigma(A) : A \in \mathcal{A}_k \\ &\geq \max\{\rho(A) : A \in \mathcal{A}_k\} \end{aligned}$$

since $||A|| \geq \rho(A)$ from [1, Theorem 5.6.9]. Hence

$$\sigma(\mathcal{A}_k) \geq \rho(\mathcal{A}_k) \geq 0; \quad \forall k \geq 1$$

and $\rho(\mathcal{A}_k) \to 0$ as $k \to \infty$. ■

(3) → (1)

From [7, Sec. 7.2], $\rho(A)$ is a continuous function of the elements of A. Since $\rho(\mathcal{A}_k) \to 0$ as $k \to \infty$, we have

$$\rho(A(k)) \to 0 \quad \text{as} \quad k \to \infty; \quad \forall A(k) \in \mathcal{A}_k.$$

Since $\rho(A(k))$ is a continuous function of $A(k)$

$$A(k) \to 0 \quad \text{as} \quad k \to \infty; \quad \forall A(k) \in \mathcal{A}_k$$

hence arriving at (6), which is the required result. ■

The following theorem illustrates the equivalence of stability of a fuzzy system and that of the corresponding switching system.

Theorem 3.2: A necessary and sufficient condition for the stability as in Definition 1 of fuzzy system (2) is that the corresponding switching system (3) be stable, as in Definition 2.

Proof necessity: The fuzzy system in (2) degenerates into the switching system when $w_i = 1$ or 0, $i = 1 \cdots N$, and $\Sigma_{i=1}^N w_i = 1$. Thus, the switching system should necessarily be stable. ■

Sufficiency: Let the switching system represented by $\mathcal{A} = \{A_1, \cdots, A_n\}$ be stable as in Definition 2. The proof that fuzzy system is also stable uses the fact that given any two sequences a_k and b_k in R^n, such that

$$\begin{aligned} &||a_k|| \geq ||b_k|| \; \forall k \geq K \quad \text{and} \quad a_k \to 0 \quad \text{as} \quad k \to \infty \\ &\text{implies } b_k \to 0 \quad \text{as} \quad k \to \infty. \end{aligned} \tag{8}$$

Now, the fuzzy system is stable if $\forall \overline{x}(0) \in \mathcal{R}^n$ and all fuzzy sets $S_i^j, i = 1 \cdots N, j = 1 \cdots n$, as $k \to \infty$

$$\overline{x}(k) \to 0$$

$$\left(\sum_{i_k=1}^{N} \alpha_{i_k}(k-1)A_{i_k}\right) \cdots \left(\sum_{i_1=1}^{N} \alpha_{i_1}(0)A_{i_1}\right)\overline{x}(0) \to 0$$

$$\text{i.e.,} \left(\sum_{i_k=1}^{N} \alpha_{i_k}(k-1)A_{i_k}\right) \cdots \left(\sum_{i_1=1}^{N} \alpha_{i_1}(0)A_{i_1}\right) \to 0$$

$$\text{i.e.} \sum_{i_1 \cdots i_k = 1 \cdots N} \alpha_{i_k}(k-1) \cdots \alpha_{i_1}(0) A_{i_k} \cdots A_{i_1} \to 0.$$

Now

$$\begin{aligned}
&\left|\left| \sum_{i_1 \cdots i_k = 1 \cdots N} \alpha_{i_1}(0) \cdots \alpha_{i_k}(k-1) A_{i_k} \cdots A_{i_1} \right|\right|_{\text{sp}} \\
&\quad \leq \sum_{i_1 \cdots i_k = 1 \cdots N} \alpha_{i_1}(0) \cdots \alpha_{i_k}(k-1) ||A_{i_k} \cdots A_{i_1}||_{\text{sp}} \\
&\quad = \sum_{i_1 \cdots i_k = 1 \cdots N} \alpha_{i_1}(0) \cdots \alpha_{i_k}(k-1) \sigma(A_{i_k} \cdots A_{i_1}) \\
&\quad \leq \sum_{i_1, \cdots, i_k = 1 \cdots N} \alpha_{i_1}(0) \cdots \alpha_{i_k}(k-1) \\
&\qquad \times \max_{i_1, i_2, \cdots, i_k} \{\sigma(A_{i_k} A_{i_{k-1}} \cdots A_{i_1})\} \\
&\quad = \sigma(\mathcal{A}_k) \sum_{i_1, \cdots, i_k = 1 \cdots N} \alpha_{i_1}(0) \cdots \alpha_{i_k}(k-1) \\
&\quad = \sigma(\mathcal{A}_k)
\end{aligned}$$

since $\Sigma_{i_j = 1 \cdots N}\, \alpha_{i_j} = 1; \forall j = 1 \cdots k$.

Now, since the switching system is stable, by Theorem 3.1 we have $\sigma(\mathcal{A}_k) \to 0$ as $k \to \infty$. From (8), left-hand side $\to 0$ and the condition for stability (4) is satisfied. Hence, the fuzzy system too is stable, as in Definition 1. ■

From now on only the stability of the switching system is mentioned. The following theorem provides a necessary condition for the stability of the switching system.

Theorem 3.3: A necessary condition for stability (as in Definition 2) of the switching system in (3) is that every finite product sequence of the matrices in $\mathcal{A}$ be stable, i.e., their spectral radius is less than one. Equivalently

$$\rho(\mathcal{A}_k) < 1; \quad \forall k \geq 1. \tag{9}$$

Proof: Suppose not. Then, $\exists A_l = A_{i_1} A_{i_2} \cdots A_{i_l} \in \mathcal{A}_l$ such that $\rho(A_{i_1} A_{i_2} \cdots A_{i_l}) \geq 1$. Then, consider the switching sequence

$$\overline{x}(k+l) = A_{i_1} A_{i_2} \cdots A_{i_l} \overline{x}(k).$$

Then, $\overline{x}(k + m * l) = A_l^m \overline{x}(k)$. Clearly, $\overline{x}(k) \not\to 0$ as $k \to \infty$ since $\rho(A_l) \geq 1$. Hence, every finite product sequence of the matrices has to be stable. ■

Comment 1: As an immediate consequence of the preceding theorem, the matrices $A_1, A_2, \cdots, A_n$ should, themselves, be necessarily stable (i.e., $\rho(A_i) < 1\ \forall i = 1 \cdots N$).

Comment 2: Also, the result expressed in [3, Theorem 4.3], a necessary condition for stability that the matrices be pairwise stable, is evident as a special case of Theorem 3.3 in which $\rho(\mathcal{A}_2)$ is considered.

IV. A Sufficient Condition for Stability

The following Theorem 4.1 provides a sufficient condition for stability of (3). Theorem 4.2 shows that when the condition in Theorem 4.1 is satisfied, there exists a common Lyapunov function $V(\overline{x})$ for all the system matrices in $\mathcal{A}$. However, this common $V(\overline{x})$ is not necessarily quadratic.

Theorem 4.1 A sufficient condition for stability, as in Definition 2 of the switching system (3), is that there exists a similarity transformation $S \in R^{n \times n}$ and a matrix norm $|| \cdot ||$ such that

$$||S^{-1}AS|| < 1; \quad \forall A \in \mathcal{A}. \tag{10}$$

Proof: Let

$$\eta = \max\{||S^{-1}AS||; \quad \forall A \in \mathcal{A}\} < 1. \tag{11}$$

Let

$$A(k) \in \mathcal{A}_k = A_{i_1} A_{i_2} \cdots A_{i_k}. \tag{12}$$

Also, let S be the similarity transformation. Now $\forall A(k) \in \mathcal{A}_k$

$$\begin{aligned}
\rho(A(k)) &= \rho(A_{i_1} A_{i_2} \cdots A_{i_k}) \\
&= \rho(S^{-1} A_{i_1} A_{i_2} \cdots A_{i_k} S) \\
&\leq ||S^{-1} A_{i_1} A_{i_2} \cdots A_{i_k} S|| \\
&\qquad \text{from Theorem 5.6.9 in [1]} \\
&\leq ||S^{-1} A_{i_1} S|| \cdot ||S^{-1} A_{i_2} S|| \cdots ||S^{-1} A_{i_k} S|| \\
&\leq \eta^k \text{ from (11) and.(12)}
\end{aligned}$$

Hence, $\rho(\mathcal{A}_k) \leq \eta^k; \eta < 1$ and $\rho(\mathcal{A}_k) \to 0$ as $k \to \infty$. Hence, the switching system is asymptotically stable. ■

Theorem 4.2: If (10) in Theorem 4.1 is satisfied, and the norm in (10) is an induced norm, then there exists a common Lyapunov function for all the system matrices in $\mathcal{A}$, i.e., there exists a function $V: R^n \to R^+$ such that $\forall A \in \mathcal{A}$:

- $V(\overline{x}(k)) > 0, \overline{x}(k) \neq 0$.
- $\Delta V(\overline{x}(k)) = V(A\overline{x}(k)) - V(\overline{x}(k)) < 0\ \forall k \geq 0$.

Proof: Let the premise in Theorem 4.1 be satisfied and S be the similarity transformation. Let $Q = S^{-1}$. Define $V(\overline{x}(k)) = ||Q\overline{x}(k)||^2$ where the vector norm here induces the matrix norm in (10). Now, since S is nonsingular, $V(\overline{x}(k)) \neq 0$ when $\overline{x}(k) \neq 0$. Now, $\forall A \in \mathcal{A}$ and $\overline{x}(k) \neq 0$,

$$\begin{aligned}
\Delta V(\overline{x}(k)) &= ||Q\overline{x}(k+1)||^2 - ||Q\overline{x}(k)||^2 \\
&= ||QA\overline{x}(k)||^2 - ||Q\overline{x}(k)||^2 \\
&= ||QAS\overline{y}(k)||^2 - ||\overline{y}(k)||^2 \\
&\qquad \text{where } \overline{y}(k) = Q\overline{x}(k) \\
&\leq ||S^{-1}AS||^2 ||\overline{y}(k)||^2 - ||\overline{y}(k)||^2 \\
&< 0 \text{ since } \overline{x}(k) \neq 0 \text{ and from.(10)}
\end{aligned}$$

This function is the common Lyapunov function. ■

However, this common Lyapunov function need not be quadratic, i.e., $V(\overline{x}(k))$ need not be of the form $\overline{x}(k)^T P \overline{x}(k)$ where P is a positive definite matrix.

Comment 1: A convenient matrix norm that can be used in (10) is the spectral norm $||\cdot||_{\rm sp} = \sigma(\cdot)$ and is induced from the Euclidean vector norm. (see [1, 5.5.6]). Hence, in particular, Theorem 4.1 reduces to the existence of a nonsingular S such that $\sigma(S^{-1}AS) < 1; \forall A \in \mathcal{A}$. In this case, the common Lyapunov function in Theorem 4.2 can be chosen to be quadratic. This is expressed in the following theorem.

Theorem 4.3: The satisfaction of (10) in Theorem 4.1 and the norm in (10) being the spectral norm (i.e., induced from the Euclidean vector norm) is a necessary and sufficient condition for the existence of a common quadratic Lyapunov function for all the system matrices in $\mathcal{A}$, i.e., $V(\overline{x})$ is of the form $\overline{x}^T P \overline{x}$ where P is a positive definite matrix.

Proof necessity: Suppose there exists a common Lyapunov function $V(\overline{x}) = \overline{x}^T P \overline{x}$ for all the system matrices $A \in \mathcal{A}$. Since P is positive definite $\exists S$ nonsingular $\ni P = Q^T Q$ where $Q = S^{-1}$. Then

$$\begin{aligned} V(\overline{x}(k)) &= \overline{x}(k)^T P \overline{x} \\ &= \overline{x}(k)^T Q^T Q \overline{x}(k) \\ &= ||Q\overline{x}(k)||^2 \end{aligned} \tag{13}$$

where $||\cdot||$ is the Euclidean norm.

Since $\Delta V(\overline{x}(k)) < 0, \forall \overline{x}(k) \in R^n$, from (13) we have $\forall A \in \mathcal{A}$

$$\begin{aligned} ||Q\overline{x}(k+1)|| - ||Q\overline{x}(k)|| &< 0 \\ ||QA\overline{x}(k)|| - ||Q\overline{x}(k)|| &< 0 \\ ||QAS\overline{y}(k)|| - ||\overline{y}(k)|| &< 0 \end{aligned} \tag{14}$$

where $\overline{y}(k) = Q\overline{x}(k)$.

Since (14) is valid $\forall \overline{y}(k) \neq 0$, we have

$$\sup_{\overline{y}(k) \neq 0} \frac{||QAS\overline{y}(k)||}{||\overline{y}(k)||} - 1 < 0$$

Hence, $\sigma(S^{-1}AS) < 1 \; \forall A \in \mathcal{A}$. ■

Sufficiency: Let S nonsingular be such that $\sigma(S^{-1}AS) < 1 \quad \forall A \in \mathcal{A}$. Then, choose $P = Q^T Q$ where $Q = S^{-1}$. Clearly, P is positive definite. Then the claim is that the common quadratic Lyapunov function is $V(\overline{x}) = \overline{x}^T P \overline{x} = ||Q\overline{x}(k)||^2$. We prove the claim by showing that $\Delta V(\overline{x}(k)) < 0 \; \forall A \in \mathcal{A}$. This proof follows the same pattern as that in Theorem 4.2 by replacing the general induced norm there by the spectral norm $\sigma(\cdot)$. ■

Comment 2: It is evident that if there exists a matrix norm $||\cdot||$ such that $||A|| < 1; \forall A \in \mathcal{A}$; then, the switching system is stable, since the similarity transformation S can be taken to be the identity matrix I. A convenient norm on the matrix is the spectral norm $||\cdot||_{\rm sp}$ equal to the largest singular value of the operand matrix. In view of the theorem and comment above, a simplified sufficiency condition, thus, is that $\sigma(A) < 1; \forall A \in \mathcal{A}$ since $\sigma(\cdot)$ the spectral norm is the norm induced from the Euclidean vector norm.

Comment 3: In particular, this motivates interest in normal matrices as (for these matrices) eigen and singular values coincide. If M is normal, $\rho(M) < 1, \sigma(M) = \rho(M) < 1$, thereby satisfying the condition mentioned in the above comment. A normal matrix M is characterized by $MM^* = M^*M$ (see [1, Sec. 2.5]).

V. SIMULTANEOUS NORMALIZATION

Since the spectral radius is a norm on normal matrices and spectral radius of each individual Amatrix of the subsystems is necessarily less than unity, simultaneous transformation of $A_i, i = 1 \cdots N$ into normal matrices through a similarity transformation is motivated. The following theorem is a fallout of the results in the previous section, and the proof follows from Theorem 4.1 and Comments 2 and 3 on Theorem 4.1.

Theorem 5.1 The switching system in (3) is stable if there exists nonsingular S such that $S^{-1}AS$ is normal $\forall A \in \mathcal{A}$ and the spectral radius of each matrix A is less than unity. ■

Since spectral radius of the matrices being less than unity is a necessary condition for stability, simultaneous similarity transformation of $A_i, i = 1 \cdots N$ into normal matrices is considered. A matrix A is said to be normalizable if there exists nonsingular S such that $S^{-1}AS$ is normal. From [5], S is necessarily of the form $S = T_A U$ where T_A is a modal matrix of A and U is a unitary matrix. Let $\mathcal{N}_A = \{S^{-1}AS : S = T_A U; U \text{ unitary}\}$. It can be shown that if $M \in \mathcal{N}_A$ then

$$\mathcal{N}_A = \{U^* M U : U \text{ is unitary}\}. \tag{15}$$

Evidently, $\rho(A) = \rho(M); M \in \mathcal{N}_A$. To check for simultaneous normalization of the matrices $A_i, i = 1 \cdots N$, initially, the problem of pairwise normalization of these matrices is considered. Once pairwise normalization is achieved, further conditions can be attached so that simultaneous normalization is achieved.

From now on, for simplicity, let the matrices $A_i, i = 1 \cdots N$ have distinct eigen values. Then the modal matrix T_A of A can be represented as $\hat{T}_A K_A M_A$ where $\hat{T}_A$ is the modal matrix of A with normalized columns of right eigen vectors of A, K_A is a diagonal matrix with nonzero complex elements and is the scaling term and M_A is a permutation matrix. Also, $M_A M_A^T = M_A^T M_A = I$. From [5, Theorem 4 and Corollary 1] we have the following.

Theorem 5.2: Let A and B be two normalizable matrices. Then they are simultaneously normalizable iff $T_A T_A^* = T_B T_B^*$. If A and B have distinct eigen values, then the above condition reduces to the existence of two positive definite matrices D_1 and D_2 such that

$$\begin{aligned} D_1 &= Q D_2 Q^* \quad Q = \hat{T}_A^{-1} \hat{T}_B \\ D_1 &= K_A K_A^*; \quad D_2 = K_B K_B^*. \end{aligned} \tag{16}$$

■

Now a relationship is developed between pairwise normalization of the matrices $A_i, i = 1 \cdots N$ and their simultaneous normalization. Let the matrices be pairwise normalizable and let the transformation $S_{ij}, i \neq j$ simultaneously normalize the matrices A_i and $A_j, \forall i, j = 1 \cdots N$. Also, $S_{ij} = S_{ji}, \forall i, j = 1 \cdots N$. The following theorem now gives the additional constraints on the transformations so that the matrices are simultaneously normalizable.

Theorem 5.3: A necessary and sufficient condition for simultaneous normalization of the matrices $A_i, i = 1 \cdots N$ is that the matrices should be pairwise normalizable and there exist fixed l_1, l_2 in $1 \cdots N$ such that

$$\begin{aligned} & S_{l_1 l_2}^{-1} S_{l_1 i} \quad \text{or} \quad S_{l_1 l_2}^{-1} S_{l_2 i} \\ & \quad = U_i \text{ a unitary matrix} \quad \forall i = 1 \cdots N. \end{aligned} \tag{17}$$

Proof: If the matrices are simultaneously normalizable say by a transformation W then clearly they are pairwise normalizable by the same transformation and, also, $WW^{-1} = I$ is unitary. Hence, only sufficiency is to be proved.

The claim is that the transformation $S_{l_1 l_2}$ simultaneously normalizes all the matrices. Now, since $S_{l_1 i}$ and $S_{l_2 i}$ normalize A_i, we have $\forall i = 1 \cdots N$

$$\begin{aligned} S_{l_1 l_2}^{-1} A_i S_{l_1 l_2} &= S_{l_1 l_2}^{-1} S_{l_1 i} M_{A_i}^1 S_{l_1 i}^{-1} S_{l_1 l_2} \quad \text{or} \\ & \qquad S_{l_1 l_2}^{-1} S_{l_2 i} M_{A_i}^2 S_{l_2 i}^{-1} S_{l_1 l_2} \\ &= U_i M_{A_i}^1 U_i^{-1} \quad \text{or} \quad U_i M_{A_i}^2 U_i^{-1} \text{ from (17)} \\ &= \tilde{M}_{A_i} \in \mathcal{N}_{A_i} \text{ from (15)} \end{aligned}$$

where $M_{A_i}^1, M_{A_i}^2 \in \mathcal{N}_{A_i}$. ■

The condition in Theorem 5.3 involves checking for $S_1^{-1} S_2$ being unitary, where S_1, S_2 are two transformations that normalize the same matrix. The following theorem provides conditions on the transformations S_1 and S_2 for this to occur.

Theorem 5.4 Let S_1 and S_2 be two transformations that normalize a matrix A. Let $S_i = \hat{T}_A K_i M_i U_i, i = 1, 2$. Then, $S_1^{-1} S_2$ is unitary iff

$$|(K_1)_{jj}| = |(K_2)_{jj}|; \quad \forall j = 1 \cdots n. \tag{18}$$

Proof: Now

$$\begin{aligned} S_1^{-1} S_2 &= (\hat{T}_A K_1 M_1 U_1)^{-1} (\hat{T}_A K_2 M_2 U_2) \\ &= U_1^* M_1^T D M_2 U_2 \end{aligned}$$

where $D = K_1^{-1} K_2$. Hence, we have

$$\begin{aligned} & (S_1^{-1} S_2)(S_1^{-1} S_2)^* \\ & \quad = (U_1^* M_1^T D M_2 U_2)(U_1^* M_1^T D M_2 U_2)^* \\ & \quad = U_1^* M_1^T D D^* M_1 U_1 \\ & \quad = \tilde{U}_1^* D D^* \tilde{U}_1 \end{aligned}$$

since M_1 is a permutation matrix. Now, $\tilde{U}_1^* D D^* \tilde{U}_1 = I$ iff D is also unitary or $DD^* = I$. Since $D = K_1^{-1} K_2, K_1, K_2$ diagonal, the condition that D be unitary reduces to the one in (18). ■

The procedure of verifying the satisfiability of Theorem 5.1 is summarized below in two steps. Step 1) checks for pairwise normalization of the matrices as in [5], and Step 2) which is carried out upon the success of Step 1) checks for additional constraints on the similarity transformations S_{ij} so that simultaneous normalization of the matrices is achieved.

Step 1: The matrices $A_i, i = 1 \cdots N$ are checked for pairwise normalization. This is done as in [5]. The conditions for pairwise normalization of two matrices, say, A_{l_1} and A_{l_2} are exposited below. The condition (16) in Theorem 5.2 is to be verified, i.e., $K_{A_{l_1}}$ and $K_{A_{l_2}}$ are to be found such that $T_{A_{l_1}} T_{A_{l_1}}^* = T_{A_{l_2}} T_{A_{l_2}}^*$. As in Theorem 5.2, define $Q = \hat{T}_{A_{l_1}}^{-1} \hat{T}_{A_{l_2}}$. Compute the matrices $Q = [q_{km}]$ and $Q^{*-1} = [s_{km}]$. With $K_{A_{l_1}} = \text{diag}\{a_k\}, K_{A_{l_2}} = \text{diag}\{b_k\}, D_1 = \text{diag}\{|a_k|^2\}$ and $D_2 = \text{diag}\{|b_k|^2\}$, as in Theorem 5.2, the condition $D_1 = Q D_2 Q^*$ is valid iff

$$|a_k|^2 s_{km} = q_{km} |b_m|^2 \quad \forall k, m = 1 \cdots n. \tag{19}$$

Now define the matrix $R_{l_1 l_2} = [r_{km}]$ for every nonzero q_{km}

$$r_{km} = \frac{s_{km}}{q_{km}}. \tag{20}$$

If the values undetermined from (20) can be adjusted such that $R_{l_1 l_2}$ has rank one, then the matrices A_{l_1} and A_{l_2} are simultaneously normalizable. ■

Remark 1: A necessary condition for (19) to be satisfied is that Q and Q^{*-1} show the same zero-nonzero pattern and all nonzero entries satisfy argument $(s_{km}) = \arg(q_{km})$.

Remark 2: If (19) is satisfied, then $R_{l_1 l_2}$ can be written as

$$r_{km} = \frac{|b_m|^2}{|a_k|^2}, \quad \forall k, m = 1 \cdots n. \tag{21}$$

Hence, $R_{l_1 l_2}$ is the outer product of two vectors whose kth elements are $|b_k|^2 = |(K_{A_{l_2}})_{kk}|^2$ and $1/|a_k|^2 = 1/|(K_{A_{l_1}})_{kk}|^2$. Hence, $R_{l_1 l_2}$ is a positive rank one matrix. Also, if $R_{l_1 l_2}$ is a rank one positive matrix then the system of equations in (19) have a solution.

Remark 3: Thus, if all the matrices are pairwise normalizable, let the transformation that pairwise normalizes A_i and A_j be

$$S_{ij} = \hat{T}_{A_i} K_i(ij) M_i(ij) U_i(ij) = \hat{T}_{A_j} K_j(ij) M_j(ij) U_j(ij).$$

Then, as constructed in Step 1, R_{ij} corresponds to the outer product of the vectors whose kth elements are $1/|(K_i(ij))_{kk}|^2$ and $|(K_j(ij))_{kk}|^2$.

Remark 4: Equation (19) represents an overdetermined set of equations and does not uniquely determine the scaling matrices $K_i(ij)$ and $K_j(ij)$ to the extent of a scaling factor in the transformations S_{ij} and, it is therefore necessary to check the conditions of Theorem 5.2 through the matrices R_{ij}.

Step 2: Once Step 1 is satisfied, the existence of the matrices S_{ij} is ensured and the matrices R_{ij} are determined as in Step 1. This step now checks for the satisfiability of (17) in Theorem 5.3. From Theorem 5.4 the condition (17) simplifies to the following:

$$\begin{aligned} \forall i = 1 \cdots N, \exists l_1, l_2 \ni |K_{l_1}(l_1 l_2)| = |K_i(l_1 i)| \quad \text{or} \\ |K_{l_2}(l_1 l_2)| = |K_i(l_2 i)|. \end{aligned}$$

As in Remark 3 above, only the outer product of vectors formed out of these scaling matrices $K_i(ij)$ and $K_j(ij)$ is expressed as matrices R_{ij}. Hence, this condition simplifies to the following:

$$\begin{aligned} & \forall i = 1 \cdots N, \exists l_1, l_2 \ni \text{the pair } (R_{l_1 l_2}, R_{l_1 i}) \\ & \qquad (R_{l_2 l_1}, R_{l_2 i}) \text{ has a common premultiplying} \\ & \qquad \text{vector in the outer product representation.} \end{aligned} \tag{22}$$

This common premultiplying vector has the jth element either $|(K_{l_1}(l_1 l_2))_{jj}|^2$ or $|(K_{l_2}(l_1 l_2))_{jj}|^2$ depending upon which of the two matrix pairs $(R_{l_1 l_2}, R_{l_1 i})$ or $(R_{l_2 l_1}, R_{l_2 i})$ satisfies (22). Appendix A provides a condition on two rank one matrices P, Q which have a common premultiplying vector in their vector outer product representation. The Lemma and the remark that follows it in the Appendix can then be used to detect the existence of l_1, l_2 such that (22) is satisfied. ■

The example below illustrates the application of both the Steps 1 and 2 in checking for simultaneous normalization of the matrices $A \in \mathcal{A}$.

Example 5.1: Let $N = n = 3$. Also let the matrices in $\mathcal{A}$ be

$$A_1 = \begin{bmatrix} -1.9 & 0.3 & 0.5 \\ -4.8 & 1.1 & 1.0 \\ -7.2 & 2.6 & 0.3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 4.7 & -1.9 & -0.1 \\ -11.1 & -4.8 & 0.1 \\ -12.3 & -5.8 & 0.1 \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} 1.7 & -0.433 & -0.033 \\ 2.2 & -0.2 & -0.1 \\ 1.4 & -0.5 & 0.6 \end{bmatrix}.$$

It can be seen that $\rho(A_1) = 0.781; \rho(A_2) = 0.985; \rho(A_3) = 0.728$ and are less than unity. Also, $\sigma(A_1) = 9.327; \sigma(A_2) = 18.882; \sigma(A_3) = 3.177$ are all greater than unity and, hence, the simplified sufficient condition in Comment 2 to Theorem 4.1 is not applicable. Now, pairwise normalization of these matrices is checked as in Step 1.

Step 1: The modal matrices of A_1, A_2, A_3 are, respectively

$$\hat{T}_{A_1} = \begin{bmatrix} 0.229 & 0.33\underline{/0.689} & 0.333\underline{/-1.69} \\ 0.688 & 0.667\underline{/1.689} & -.667\underline{/-1.689} \\ 0.688 & 0.667\underline{/2.33} & 0.667\underline{/-2.33} \end{bmatrix}$$

$$\hat{T}_{A_2} = \begin{bmatrix} 0.224\underline{/0.561} & 0.224\underline{/-0.561} & 0.408 \\ 0.57\underline{/0.364} & 0.57\underline{/-0.364} & 0.817 \\ 0.791\underline{/0.703} & 0.791\underline{/-0.703} & 0.408 \end{bmatrix}$$

and

$$\hat{T}_{A_3} = \begin{bmatrix} 0.341\underline{/-1.82} & 0.341\underline{/1.18} & 0.218 \\ 0.763\underline{/-1.997} & 0.763\underline{/1.997} & 0.436 \\ 0.55\underline{/-2.337} & 0.55\underline{/2.337} & 0.873 \end{bmatrix}.$$

To check for pairwise normalization of A_1, A_2, we find $Q_{12} = \hat{T}_{A_1}^{-1}\hat{T}_{A_2}$ and Q_{12}^{*-1}, respectively, as

$$\begin{bmatrix} 0.689\underline{/-0.225} & 0.689\underline{/0.225} & 0 \\ 0.53\underline{/-1.45} & 0.53\underline{/2.14} & 0.685\underline{/-1.226} \\ 0.53\underline{/-2.14} & 0.53\underline{/1.45} & 0.685\underline{/1.226} \end{bmatrix}$$

$$\begin{bmatrix} 0.726\underline{/-0.225} & 0.726\underline{/0.225} & 0 \\ 0.471\underline{/-1.45} & 0.471\underline{/2.14} & 0.73\underline{/-1.226} \\ 0.471\underline{/-2.14} & 0.471\underline{/1.45} & 0.73\underline{/1.226} \end{bmatrix}.$$

Hence, the matrix R_{12} computed as in (20) and the entry r_{13} not found from (20) is adjusted such that R_{12} has unit rank is

$$R_{12} = \begin{bmatrix} 1.053 & 1.053 & 1.263 \\ 0.889 & 0.889 & 1.067 \\ 0.889 & 0.889 & 1.067 \end{bmatrix}.$$

Similarly, $Q_{23} = \hat{T}_{A_2}^{-1}\hat{T}_{A_3}$ and $Q_{31} = \hat{T}_{A_3}^{-1}\hat{T}_{A_1}$ are computed. Also, $Q_{21} = Q_{12}^{-1}, Q_{13} = Q_{31}^{-1}, Q_{32} = Q_{23}^{-1}$ are computed. As before, the matrices R_{23}, R_{21}, R_{32} and R_{31} are calculated and the matrices R_{13} and R_{23} are

$$R_{13} = \begin{bmatrix} 1.132 & 1.132 & 1.105 \\ 0.956 & 0.956 & 0.933 \\ 0.956 & 0.956 & 0.933 \end{bmatrix} \quad \text{and}$$

$$R_{23} = \begin{bmatrix} 1.075 & 1.075 & 1.05 \\ 1.075 & 1.075 & 1.05 \\ 0.896 & 0.896 & 0.875 \end{bmatrix}.$$

It is seen that the matrices R_{12}, R_{23}, R_{13} are of rank one and, hence, the matrices A_1, A_2, A_3 are pairwise normalizable. Also, the matrices R_{21}, R_{32}, R_{31} are also of rank one.

Step 2: It is seen that the pair (R_{12}, R_{13}) has a common premultiplying vector in their vector outer product representation. Thus, $l_1 = 1, l_2 = 2$ can be chosen and (22) is satisfied. Thus, the matrices are simultaneously normalizable and, hence, the corresponding switching system is stable.

Comment 4: Simultaneous normalization can be thought of as an extension of simultaneous diagonalization of the matrices since diagonal matrices are normal. This case has been studied in [6] as a class of commuting matrices and the proof there proceeds by actually constructing a common Lyapunov matrix. We show that this result follows naturally in our framework of simultaneously normalizable matrices. This ties the circle of ideas of demonstrating stability by showing the existence of or by actually constructing a common quadratic Lyapunov function for the matrices in $\mathcal{A}$.

Theorem 5.5 [6]: If the matrices $A_i, i = 1 \cdots N$ commute pairwise, then the switching system (3) is stable.

Proof: Let the matrices $A_1, A_2, \cdots A_n$ commute pairwise. Then the matrices are simultaneously diagonalizable [1, Theorem 1.3.12] and, hence, simultaneously normalizable. The switching system then is evidently stable by Theorem 5.1.

To see this, the matrices are now pairwise diagonalizable [1, Theorem 1.3.12] and let S_{ij} be the transformation which diagonalizes both A_i and A_j. Here, $S_{ij} = S_{ji}$. The claim is that any of these S_{ij} will simultaneously diagonalize all the matrices $A_i, i = 1 \cdots N$. This is seen as shown below.

Now, if S_1, S_2 diagonalize a matrix A, and D is any diagonal matrix, then $S_1^{-1}S_2DS_2^{-1}S_1 = D$. This is because

$$\begin{aligned} S_1^{-1}S_2 &= (\hat{T}_A K_1 M_1)^{-1}(\hat{T}_A K_2 M_2) \\ &= M_1^T D_{12} M_2, \quad \text{where} \\ D_{12} &= K_1^{-1}K_2 \text{ is diagonal.} \end{aligned}$$

Now

$$\begin{aligned} S_1^{-1}S_2DS_2^{-1}S_1 &= M_1^T D_{12} M_2 D M_2^T D_{12}^{-1} M_1 \\ &= M_1^T D_{12} D D_{12}^{-1} M_1 \\ &= D. \end{aligned} \tag{23}$$

Here, the notation is as previously used. $\hat{T}_A$ is the modal matrix of A with normalized right eigen vectors of A as its columns, K_1, K_2 are diagonal scaling matrices with nonzero complex entries, and M_1, M_2 are permutation matrices. Now, since $S_{l_1 i}$ diagonalizes A_i to say D_i, we have $\forall i = 1 \cdots N$,

$$\begin{aligned} S_{l_1 l_2}^{-1} A_i S_{l_1 l_2} &= S_{l_1 l_2}^{-1} S_{l_1 i} D_i S_{l_1 i}^{-1} S_{l_1 l_2} \\ &= D_i \text{ from(23).} \end{aligned}$$

Hence, any transformation $S_{l_1 l_2}$ that diagonalizes A_{l_1} and A_{l_2} will also diagonalize the other matrices. Hence, the matrices are simultaneously diagonalizable. ∎

VI. CONCLUSION

Stability of the Sugeno–Takagi model [2] of a fuzzy system has been considered in the asymptotic sense. A necessary condition for the global asymptotic stability which generalizes a result reported in [3] is given. A sufficient condition is also provided here for the global asymptotic stability of the system. A particular case of this sufficient condition, simultaneous normalization of the system matrices, is considered and a constructive procedure to check for simultaneous normalization is developed. This condition is easily checked using software utilities such as Matlab. A necessary and sufficient condition for asymptotic stability is yet elusive and further efforts should be aimed in this direction. The approach considered in this paper concentrates only on the consequents of the fuzzy implication rules and brackets all antecedents together, i.e., these results hold for very general systems since stability is shown for all possible fuzzy sets and membership functions. A method to take into account the specific knowledge of antecedents is needed.

APPENDIX A

Given two rank one matrices P and Q, a procedure is outlined below which checks whether the matrices have a common premultiplying vector in their vector outer product representation.

Lemma 1: Let $P = [p_{ij}]$ and $Q = [q_{ij}]$ be rank one matrices. Then necessary and sufficient conditions on the matrices so that they may have a common premultiplying vector in their vector outer product representation are

1) the zero-nonzero pattern of P and Q coincide. Upon a nonmatch, the column containing the zero that did not match should all be zero;
2) $\forall j = 1 \cdots n$:

$$\forall i = 1 \cdots n, \quad q_{ij} \neq 0, \quad \frac{p_{ij}}{q_{ij}} = k_j.$$

Remark 1: If P and Q have a common premultiplying vector in their vector outer product representation, then $\exists a, b, c \in R^n \ni P = ab^T$ and $Q = ac^T$. The two conditions in the lemma are evident from this representation.

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