Optimal Sequences and Sum Capacity of Synchronous CDMA Systems

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Abstract—The sum capacity of a multiuser synchronous CDMA system is completely characterized in the general case of asymmetric user power constraints—this solves the open problem posed in [7] which had solved the equal power constraint case. We identify the signature sequences with real components that achieve sum capacity and indicate a simple recursive algorithm to construct them.

Index Terms—Capacity region, CDMA, optimal signature sequences, sum capacity.

I. INTRODUCTION

AN important multiple-access technique in wireless networks and other common channel communication systems is Code-Division Multiple Access (CDMA). Each user shares the entire bandwidth with all the other users and is distinguished from the others by its *signature sequence* or *code*. Each user spreads its information on the common channel through modulation using its signature sequence. Then, the receiver demodulates the transmitted messages upon observing the sum of the transmitted signals embedded in noise. We focus on symbol-synchronous CDMA (S-CDMA) systems where in each symbol interval the received signal is the sum of the transmitted signals in that symbol interval alone embedded in additive white Gaussian noise.

Of fundamental interest in this system is the capacity region defined as the set of information rates at which users can transmit while retaining reliable transmission. This problem was addressed in [8] and the capacity region was characterized as a function of the signature sequences and average input power constraints of the users. However, the choice of the signature sequences of the users is left open to the designer of the CDMA system and it was suggested in [8] that the signature sequences could be optimized given the constraints of the problem. We address this issue and focus on finding the "sum capacity" (maximum sum of the achievable rates of all users per unit processing gain; maximum over all choices of signature sequences).

This problem has been attempted in [7] where the authors derive an upper bound on the sum capacity. This upper bound is $\frac{1}{2} \log \left(1 + \frac{p_{\text{tot}}}{\sigma^2}\right)$, the capacity of the system with "no spreading," i.e., of the system with processing gain 1

and an appropriate power constraint on its input (p_{tot}, sum of power constraints of the users) and σ^2 is the variance of the additive white Gaussian noise. This result assumed that the signature sequences had real components (as opposed to components in $\{+1, -1\}$; the problem of characterizing sum capacity with this constraint remains open) and we retain this assumption in this paper. It turns out that this upper bound on sum capacity is not achievable for all values of the average input power constraints of the users. In this paper, we completely characterize sum capacity of the S-CDMA channel. We identify the sequences (as a function of the average input power constraints of the users) that achieve sum capacity (called *optimal* sequences), and discuss some algorithmic ways to construct them. Our main result, the complete characterization of sum capacity, allows us to conclude.

1) A user is said to be *oversized* if its input power constraint is large *relative* to the input power constraints of the other users. The optimal signature sequence allocation is to allocate orthogonal sequences (hence independent channels) to oversized users.
2) Nonoversized users are allocated sequences that we denote *generalized Welch-Bound-Equality (WBE) sequences*.
3) Sum capacity is equal to $\frac{1}{2} \log \left(1 + \frac{p_{\text{tot}}}{\sigma^2}\right)$ (the upper bound derived in [7]) if and only if no user is oversized.

This paper is organized as follows: We discuss the model of the S-CDMA system briefly and recall the characterization of the capacity region for fixed choice of signature sequences in Section II. Section II also develops some notation we will require in the derivation of our main result. Our main result, the characterization of sum capacity, is in Section III. In Section IV, we outline an algorithm to construct the optimal signature sequences (namely, generalized WBE sequences). This algorithmic procedure is exemplified by a system with three users and processing gain 2. The results are summarized in Section V which also contains some concluding remarks.

II. S-CDMA MODEL AND NOTATION

A. S-CDMA Channel and Capacity Region

We consider the discrete-time, baseband S-CDMA channel model. There are K users in the system and the processing gain is N. Both K and N will be fixed throughout this paper. Since we have assumed a synchronous model we can restrict our attention to one symbol interval. As is traditional, we

model the information transmitted (symbol) by each user as independent random variables $X_1, \cdots, X_K$. We assume that there is an average input power constraint on the transmit symbols given by

$$E[X_i^2] \leq p_i, \qquad \forall i = 1 \cdots K.$$

Let $D = \text{diag}\{p_1, \cdots, p_K\}$ and the maximum total average input power be $p_{\text{tot}} = \Sigma_{i=1}^K p_i$. Let the signature sequence of user i be represented by s_i, a vector in $\mathbb{R}^N$. Each signature sequence has power equal to N, i.e., for each user i, we have $s_i^t s_i = N$. We assume an ambient white Gaussian noise, denoted by $W \sim \mathcal{N}(0, \sigma^2 I)$ independent of the transmitted symbols. Then the received signal, represented by Y, can be written as

$$Y = \sum_{i=1}^K s_i X_i + W.$$

Let us represent the $N \times K$ matrix $[s_1 \cdots s_K]$ by S. The S-CDMA channel above is a special case of the K-user Gaussian multiple access channel and the capacity region (set of rates at which reliable communication is possible) is well known (see [2, Sec. VII]) as the closure of the convex hull of the union over all product probability densities $p_{\overline{X}}$ on the inputs $X_1, \cdots, X_K$ of the rate regions

$$\mathcal{C}(S, p_{\overline{X}}) = \bigcap_{J \subseteq \{1, \cdots, K\}} \left\{ (R_1, \cdots, R_K) : 0 \leq \sum_{i \in J} R_i \leq I \right.$$
$$\left. \cdot (Y; X_i, i \in J | X_i, i \in J^c) \right\}. \quad (1)$$

Continuing as in [7], the union region over the product distributions, as a function of S, can be simply written as

$$\mathcal{C}(S) = \bigcap_{J \subseteq \{1, \cdots, K\}} \left\{ (R_1, \cdots, R_K) : 0 \leq \sum_{i \in J} R_i \leq \frac{1}{2N} \right.$$
$$\left. \cdot \log \left[\det \left(I + \frac{1}{\sigma^2} S_J D_J S_J^t \right) \right] \right\} \quad (2)$$

where J is a nonnull set and R_i is the rate in bits per chip of user i. Here, S_J is $N \times |J|$ matrix $\{s_i : i \in J\}$ and D_J is the $|J| \times |J|$ matrix $\text{diag}\{p_i : i \in J\}$. Observe that the choice of product distribution for each user i, X_i distributed as $\mathcal{N}(0, p_i)$ makes the region in (1) equal to that in (2). Sum capacity represents the maximum sum of rates of all users per unit processing gain at which users can transmit reliably. Following the notation in [7], the sum capacity is defined formally as

$$\mathcal{C}_{\text{sum}} = \max_{S \in \mathcal{S}} \max_{R \in \mathcal{C}(S)} \sum_{i=1}^K R_i \quad (3)$$

where $\mathcal{S}$ is the set of all $N \times K$ real matrices with all columns having l_2 norm equal to $\sqrt{N}$.

B. Majorization: Definition and Some Key Results

We introduce the notion of majorization and recall some key results that we require in the derivation of $\mathcal{C}_{\text{sum}}$ in Section III. We begin with some definitions.

Definition 2.1: For any $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let

$$x_{[1]} \geq \cdots \geq x_{[n]}$$

denote the components of x in decreasing order, called the *order statistics* of x.

Majorization makes precise the vague notion that the components of a vector x are "less spread out" or "more nearly equal" than are the components of a vector y by the statement x is majorized by y.

Definition 2.2: For $x, y \in \mathbb{R}^n$, say that x is majorized by y (or y majorizes x) if

$$\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}, \qquad k = 1 \cdots n - 1$$
$$\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}.$$

A comprehensive reference on majorization and its applications is [4]. A simple (trivial, but important) example of majorization between two vectors is the following:

Example 2.1: For every $a \in \mathbb{R}^n$ such that $\sum_{i=1}^n a_i = 1$

$$(a_1, \cdots, a_n) \text{ majorizes } \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n} \right).$$

It is well known that the sum of diagonal elements of a matrix is equal to the sum of its eigenvalues. When the matrix is symmetric the *precise* relationship between the diagonal elements and the eigenvalues is that of majorization:

Lemma 2.1 ([4, Theorem 9.B.1 and 9.B.2]): Let H be a symmetric matrix with diagonal elements $h_1, \cdots, h_n$ and eigenvalues $\lambda_1, \cdots, \lambda_n$ we have

$$(\lambda_1, \cdots, \lambda_n) \text{ majorizes } (h_1, \cdots, h_n).$$

That $h = (h_1, \cdots, h_n)$ and $\lambda = (\lambda_1, \cdots, \lambda_n)$ cannot be compared by an ordering stronger than majorization is the consequence of the following converse: If $h_1 \geq \cdots \geq h_n$ and $\lambda_1 \geq \cdots \lambda_n$ are $2n$ numbers such that λ majorizes h, then there exists a real symmetric matrix H with diagonal elements $h_1, \cdots, h_n$ and eigenvalues $\lambda_1, \cdots, \lambda_n$.

We will also need the following definition:

Definition 2.3: A real-valued function $\phi \colon \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-concave if for all $x, y \in \mathcal{R}^n$ such that y majorizes x we have $\phi(x) \geq \phi(y)$. Say that ϕ is strictly Schur-concave if y majorizes x and $y \neq x$ implies that $\phi(x) > \phi(y)$.

An important class of Schur-concave functions is the following ([4, Theorem 3.C.1]).

Example 2.2: If $g \colon \mathbb{R} \to \mathbb{R}$ is concave then the symmetric concave function $\phi(x) = \Sigma_{i=1}^n g(x_i)$ is Schur-concave.

III. Characterization of Sum Capacity

This section contains our main result, the characterization of $\mathcal{C}_{\text{sum}}$. In [7], an upper bound was derived for $\mathcal{C}_{\text{sum}}$, this upper bound being the sum capacity of the "unrestricted" S-CDMA channel, i.e., the situation of no spreading when $N = 1$ with the appropriate power constraint on its input (sum of the power constraints of the users given by p_{tot}). The latter channel is just the K-user Gaussian multiple-access channel and its sum capacity is

$$C = \frac{1}{2} \log \left(1 + \frac{p_{\text{tot}}}{\sigma^2} \right) \qquad (4)$$

(see [2, Ch. 7]) in bits per chip. When the input power constraints are equal, it was shown in [7] that WBE signature sequences (so called because they meet the Welch-Bound-Equality (see [11] and [5])) achieve sum capacity, and sum capacity then equals the upper bound C. However, we will show that this bound is not tight for arbitrary values of the average input power constraints of the users and that there is a strict loss in sum capacity when the power constraints are "far apart;" we make this notion precise.

When $K \leq N$, it is easy to verify that the sum capacity is achieved by signature sequences chosen orthogonal to each other and we have

$$\mathcal{C}_{\text{sum}} = \frac{1}{2N} \sum_{i=1}^{K} \log \left(1 + \frac{N p_i}{\sigma^2} \right).$$

Note that when $K = N$ and all the power constraints p_i are the same, $\mathcal{C}_{\text{sum}} = \frac{1}{2} \log \left(1 + \frac{p_{\text{tot}}}{\sigma^2} \right)$ the same as the sum capacity of the system with no spreading; this is the well-known fact that for equal-power users, orthogonal multiple access incurs no loss in capacity relative to unconstrained multiple access. When $K = N$ and there is an asymmetry in the power constraints of the users, a simple argument shows that $\mathcal{C}_{\text{sum}} < C$. When $K < N$ the claim is that $\mathcal{C}_{\text{sum}} < C$. To see this: observe that $(p_1, \cdots, p_K)$ majorizes the vector $\left(\frac{p_{\text{tot}}}{K}, \cdots, \frac{p_{\text{tot}}}{K} \right)$ (see Example 2.1). It can be verified that the map

$$(p_1, \cdots, p_K) \mapsto \frac{1}{2N} \sum_{i=1}^{K} \log \left(1 + \frac{N p_i}{\sigma^2} \right)$$

is Schur-concave (see Definition 2.3 and Example 2.2). Hence

$$\mathcal{C}_{\text{sum}} = \frac{1}{2N} \sum_{i=1}^{K} \log \left(1 + \frac{N p_i}{\sigma^2} \right)$$
$$\leq \frac{K}{2N} \log \left(1 + \frac{N p_{\text{tot}}}{K \sigma^2} \right)$$
$$< \frac{1}{2} \log \left(1 + \frac{p_{\text{tot}}}{\sigma^2} \right)$$

where in the last step we used the inequality $(1+x)^a < 1 + ax$ for $x > 0$ and $a \in (0, 1)$.

Henceforth we assume $K > N$. Say that a user is *oversized*[1] if its input power constraint is large relative to the input power

[1] We would like to thank Prof. Sergio Verdú for simplifying our presentation by suggesting this terminology.

constraints of the other users. More precisely, user i is defined to be oversized if

$$p_i > \frac{\sum_{j=1}^{K} p_j 1_{\{p_i > p_j\}}}{N - \sum_{j=1}^{K} 1_{\{p_j \geq p_i\}}}. \qquad (5)$$

Denote the set of oversized users as $\mathcal{K}$. A key observation is the following: $\mathcal{K}$ is the *unique* subset of users satisfying

$$(N - |\mathcal{K}|) \min_{i \in \mathcal{K}} p_i > \sum_{j \notin \mathcal{K}} p_j \geq (N - |\mathcal{K}|) \max_{i \notin \mathcal{K}} p_i. \qquad (6)$$

Some simple observations can now be made.

1) No user is oversized if and only if $\frac{1}{N} \sum_{i=1}^{K} p_i \geq p_j$ for every user j.

2) When all the input power constraints are equal, no user is oversized.

3) There can be at most $N - 1$ oversized users.

4) If a user i is oversized then every user with input power constraint at least p_i is also oversized.

5) A simple algorithm to find $\mathcal{K}$ is the following:
   Step 1 Start with $\mathcal{K} = \Phi$.
   Step 2 If $\sum_{j \notin \mathcal{K}} p_j \geq (N - |\mathcal{K}|) \max_{j \notin \mathcal{K}} p_j$, then terminate.
   Step 3 Else, update $\mathcal{K} = \mathcal{K} \cup \{\arg\max_{j \notin \mathcal{K}} p_j\}$.
   Step 4 Return to Step 2.

We are now ready to state our main result.

Theorem 3.1:

$$\mathcal{C}_{\text{sum}} = \frac{N - |\mathcal{K}|}{2N} \log \left(1 + \frac{N \sum_{j \notin \mathcal{K}} p_j}{(N - |\mathcal{K}|) \sigma^2} \right)$$
$$+ \frac{1}{2N} \sum_{i \in \mathcal{K}} \log \left(1 + \frac{N p_i}{\sigma^2} \right). \qquad (7)$$

Proof: By definition, from (3)

$$\mathcal{C}_{\text{sum}} = \max_{S \in \mathcal{S}} \max_{R \in \mathcal{C}(S)} \sum_{i=1}^{K} R_i$$
$$= \max_{S \in \mathcal{S}} \frac{1}{2N} \log \left[\det \left(I + \frac{1}{\sigma^2} S D S^t \right) \right]$$
$$\qquad \text{from (2), also see [7]}$$
$$= \max_{S \in \mathcal{S}} \frac{1}{2N} \sum_{i=1}^{N} \log \left(1 + \frac{N}{\sigma^2} \lambda_i(S) \right) \qquad (8)$$

where $\lambda(S) = (N\lambda_1(S), \cdots, N\lambda_N(S)) \in \mathcal{R}_+^N$ denotes the vector of eigenvalues of the matrix $S D S^t$. Define the convex set $\mathcal{N}$ in the positive orthant of $\mathbb{R}^N$ by

$$\mathcal{N} = \{(\lambda_1, \cdots, \lambda_N) \in \mathbb{R}_+^N : (\lambda_1, \cdots, \lambda_N, 0, \cdots, 0)$$
$$\text{majorizes } (p_1, \cdots, p_K)\}.$$

We first identify the region of eigenvalues of the matrix $\frac{1}{N} S D S^t$ as S varies in $\mathcal{S}$ to be exactly $\mathcal{N}$. Formally, we

claim that

$$\left\{\frac{1}{N}\lambda(S)\colon S \in \mathcal{S}\right\} = \mathcal{N}. \tag{9}$$

First consider $S \in \mathcal{S}$. Let $\tilde{\lambda}(S) \in \mathbb{R}_+^K$ be the vector of eigenvalues of the matrix $D^{\frac{1}{2}}S^tSD^{\frac{1}{2}}$ and observe that $\tilde{\lambda}(S)$ is just the vector $\lambda(S)$ with $K - N$ appended zeros. The observation that the diagonal elements of $\frac{1}{N}D^{\frac{1}{2}}S^tSD^{\frac{1}{2}}$ are $p_1, \cdots, p_K$ coupled with an appeal to Lemma 2.1, allows us to conclude that $\frac{1}{N}\lambda(S) \in \mathcal{N}$. To see the other direction, consider $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathcal{N}$. Then, by definition, the vector $(\lambda_1, \cdots, \lambda_N, 0, \cdots, 0)$ majorizes the vector $(p_1, \cdots, p_K)$. Appealing to Lemma 2.1, there exists a symmetric matrix H with eigenvalues $\lambda_1, \cdots, \lambda_N, 0, \cdots, 0$ and diagonal elements $p_1, \cdots, p_K$. Let $v_1, \cdots, v_N \in \mathbb{R}^K$ be the normalized eigenvectors of H corresponding to the eigenvalues $\lambda_1, \cdots, \lambda_N$. Let $V^t = \{v_1 v_2 \cdots v_N\}$. If we let Λ to be the diagonal matrix with entries $\lambda_1, \cdots, \lambda_N$, then $H = V^t\Lambda V$. Now define $S = \sqrt{N}\Lambda^{\frac{1}{2}}VD^{-\frac{1}{2}}$. Then, since the square of the l_2 norms of the columns of S are the diagonal elements of S^tS, we verify that $S^tS = ND^{-\frac{1}{2}}HD^{-\frac{1}{2}}$ has diagonal entries equal to N concluding that $S \in \mathcal{S}$. This completes the proof of the claim in (9).

Then the sum capacity can be rewritten as, from (8),

$$C_{\text{sum}} = \max_{\lambda \in \mathcal{N}} \frac{1}{2N}\sum_{i=1}^N \log\left(1 + \frac{N}{\sigma^2}\lambda_i\right). \tag{10}$$

The following lemma identifies a "minimal" element in $\mathcal{N}$. Recall that the set of oversized users is denoted by $\mathcal{K}$.

Lemma 3.1: Let $\lambda^* = (\lambda_1^*, \cdots, \lambda_N^*) \in \mathbb{R}_+^N$ be

$$\lambda^* = \left(\frac{\sum\limits_{j \notin \mathcal{K}} p_j}{N - |\mathcal{K}|}, \cdots, \frac{\sum\limits_{j \notin \mathcal{K}} p_j}{N - |\mathcal{K}|}, p_i; i \in \mathcal{K}\right). \tag{11}$$

Then
1) $\lambda^* \in \mathcal{N}$.
2) If $\lambda \in \mathcal{N}$ then λ majorizes λ^*.

Suppose this is true. As observed earlier, the map

$$(\lambda_1, \cdots, \lambda_N) \mapsto \frac{1}{2N}\sum_{i=1}^N \log\left(1 + \frac{N\lambda_i}{\sigma^2}\right)$$

is Schur-concave. Then (7) follows from (10) by an appeal to Lemma 3.1 above and the proof is complete. We only need to prove the lemma above.

Proof of Lemma 3.1: It is straightforward from the definition of λ^* and properties of oversized users that $\lambda^* \in \mathcal{N}$. Let $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathcal{N}$ and $\lambda_{[1]}, \cdots, \lambda_{[N]}$ denote the order statistics of λ (see Definition 2.1 for the notation). By the definition of λ^* in (11), it can be verified that the following relation is true among the elements of λ^*

$$\lambda_{[1]}^* = \max\left\{\frac{p_{\text{tot}}}{N}, p_{[1]}\right\}$$

$$\lambda_{[k+1]}^* = \max\left\{\frac{p_{\text{tot}} - \sum\limits_{i=1}^k \lambda_{[i]}^*}{N - k}, p_{[k+1]} + \sum_{i=1}^k\left(p_{[i]} - \lambda_{[i]}^*\right)\right\},$$
$$\forall k = 1 \cdots N - 1. \tag{12}$$

Hence $\forall k = 1 \cdots N - 1$ we can write

$$\sum_{i=1}^{k+1}\lambda_{[i]}^* = \max\left\{\sum_{i=1}^{k+1}p_{[i]}, \frac{p_{\text{tot}}}{N-k} + \frac{N-k-1}{N-k}\sum_{i=1}^k\lambda_{[i]}^*\right\}. \tag{13}$$

Now, since $\lambda \in \mathcal{N}$ we have $\sum_{i=1}^N \lambda_i = p_{\text{tot}}$ and hence $\lambda_{[1]} \geq \frac{p_{\text{tot}}}{N}$. Furthermore, $\lambda_{[1]} \geq p_1$. Hence

$$\lambda_{[1]} \geq \max\left\{\frac{p_{\text{tot}}}{N}, p_1\right\} = \lambda_{[1]}^*.$$

We complete the proof of the claim that λ majorizes λ^* by induction. Suppose

$$\sum_{i=1}^k \lambda_{[i]} \geq \sum_{i=1}^k \lambda_{[i]}^*$$

for some $1 \leq k < N$. Since

$$\sum_{i=1}^{N-k} \lambda_{[k+i]} = p_{\text{tot}}\sum_{i=1}^k \lambda_{[i]}$$

and

$$\lambda_{[k+1]} \geq \lambda_{[k+2]} \geq \cdots \geq \lambda_{[N]}$$

we have

$$\lambda_{[k+1]} \geq \frac{p_{\text{tot}} - \sum\limits_{i=1}^k \lambda_{[i]}}{N - k}.$$

Hence

$$\sum_{i=1}^{k+1}\lambda_{[i]} \geq \frac{p_{\text{tot}}}{N-k} + \left(\frac{N-k-1}{N-k}\right)\sum_{i=1}^k \lambda_{[i]}$$

$$\geq \frac{p_{\text{tot}}}{N-k} + \left(\frac{N-k-1}{N-k}\right)\sum_{i=1}^k \lambda_{[i]}^*$$
$$\text{by induction hypothesis.} \tag{14}$$

Since $\sum_{i=1}^{k+1}(\lambda_{[i]} - p_{[i]}) \geq 0$, from (14), we have

$$\sum_{i=1}^{k+1}\lambda_{[i]} \geq \max\left\{\Sigma_{i=1}^{k+1}p_{[i]}, \frac{p_{\text{tot}}}{N-k} + \frac{N-k-1}{N-k}\sum_{i=1}^k\lambda_{[i]}^*\right\}$$
$$= \sum_{i=1}^{k+1}\lambda_{[i]}^* \quad \text{from (13).}$$

This is true for all $k = 1 \cdots N - 1$. Hence λ majorizes λ^* and λ^* is a Schur-minimal element of $\mathcal{N}$. This completes the proof of the lemma and hence that of the theorem. $\square$

As a corollary of Theorem 3.1, we have a necessary and sufficient condition for when $\mathcal{C}_{\text{sum}}$ equals the upper bound C (in (4)).

Corollary 3.1:

$$\mathcal{C}_{\text{sum}} = C = \frac{1}{2} \log \left(1 + \frac{p_{\text{tot}}}{\sigma^2} \right)$$

if and only if no user is oversized.

The proof is obvious from Theorem 3.1. Theorem 3.1 also shows that when $p_{\text{tot}} < N p_i$ for some user i, then the users' power constraints are not "close enough" and the resulting sum capacity of the system $\mathcal{C}_{\text{sum}}$ is *strictly* smaller than C. This observation follows from the fact that the map

$$(\lambda_1, \cdots, \lambda_N) \mapsto \frac{1}{2N} \sum_{i=1}^{N} \log \left(1 + \frac{N \lambda_i}{\sigma^2} \right)$$

is strictly Schur-concave (see Definition 2.3).

IV. CONSTRUCTION OF OPTIMAL SEQUENCES

In this section we identify and provide an explicit algorithm to construct signature sequences that achieve sum capacity given by Theorem 3.1. We first focus on the case considered in Corollary 3.1. This is the situation when $\mathcal{C}_{\text{sum}}$ is equal to C, the capacity of the system with no spreading.

Following the proof of Theorem 3.1, we observe that the sequences (indicated by the matrix S) for which sum capacity is achieved are precisely those with the property $SDS^t = p_{\text{tot}}I$ where, as before, D is a diagonal matrix with diagonal entries $p_1, p_2, \cdots, p_K$. We now outline an algorithm to construct such a matrix $S \in \mathcal{S}$. Given any x majorized by y, the proof of [4, Theorem 9.B.2] (the statement of [4, Theorem 9.B.2] is contained in Lemma 2.1) indicates a recursive way of constructing a symmetric matrix with vector of diagonal entries equal to x and vector of eigenvalues equal to y. Below, we outline an algorithm (recursive) that achieves the same goal as above but appears more direct than the classical proof indicated in [4, Theorem 9.B.2]. Also, this algorithm leads directly to the construction of optimal signature sequences, i.e., construction of $S \in \mathcal{S}$ such that $SDS^t = p_{\text{tot}}I$. A somewhat related construction appears in [6]. The following notation and definitions are from [4].

A. Constructing a Symmetric Matrix with Given Diagonal Entries and Eigenvalues

A permutation matrix $Q \in \mathbb{R}^{K \times K}$ is a matrix with each entry equal to either 0 or 1 such that each row and column has exactly one entry equal to 1. A *T-transform* is a doubly stochastic matrix of the form

$$T = \alpha I + (1 - \alpha) Q$$

for some $\alpha \in \{0, 1\}$ and some permutation matrix Q with $K - 2$ diagonal entries equal to 1. To see the operation of a T-transform, let $y = (y_1, \cdots, y_K) \in \mathbb{R}^K$. Let $Q_{kl} = Q_{lk} = 1$ for some indices $k < l$. Then

$$Qy = (y_1, \cdots, y_{k-1}, y_l, y_{k+1}, \cdots, y_{l-1}, y_k, y_{l+1}, \cdots, y_K)$$

and hence

$$Ty = (y_1, \cdots, y_{k-1}, \alpha y_k + (1 - \alpha) y_l, y_{k+1}, \cdots, y_{l-1},$$
$$\alpha y_l + (1 - \alpha) y_k, y_{l+1}, \cdots, y_K).$$

The following is a fundamental result from the theory of majorization (Lemma 2.B.1 in [4]).

Lemma 4.1: If x is majorized by y then there exists a sequence of T-transforms $T_1, \cdots, T_n$ such that $x = T_n \cdots T_2 T_1 y$ and $n < K$.

For notational simplicity we shall assume the largest number of T-transforms are required (this is the worst case) and let $n = K - 1$ (there is no loss in generality, the arguments below will only have to be slightly modified to take $n < K - 1$ into account). Let $y^{(i)} = T_i y^{(i-1)}$ and $y^{(0)} = y$. Then the lemma says that $x = y^{(K-1)}$. Let $T_i = \alpha_i I + (1 - \alpha_i) Q_i$ where Q_i interchanges the k_ith and l_ith elements (say, $k_i < l_i$). The proof of [4, Lemma 2.B.1] explicitly constructs (recursively) the values of α_i, k_i and l_i thereby completely specifying T_i for each $i = 1 \cdots K - 1$. An inspection of the same proof (reproduced in the Appendix for completeness) also shows that

$$k_{i-1} \le k_i < l_i \le l_{i-1} \quad \text{and} \quad k_i - k_{i-1} + l_{i-1} - l_i \ge 1. \tag{15}$$

Define U_i as

$$U_i(m, n) = \begin{cases} \sqrt{T_i(m, n)}, & \text{if } m \le n \\ -\sqrt{T_i(m, n)}, & \text{else.} \end{cases}$$

Note that $U_i(m, m) = 1$ for all $m \ne k_i, l_i$. Also, $U_i(m, n) = 0$ for $m \ne k_i$ and $n \ne l_i$ and $m < n$. Hence U_i is a unitary matrix. Let $H_0 = \text{diag}\{y_1, \cdots, y_K\}$ and $H_i = U_i^t H_{i-1} U_i$. Now consider the following claim for each $i = 0 \cdots K - 1$:

$$y^{(i)} \text{ is the vector of diagonal entries of } H_i. \tag{16}$$

Suppose that this is true. Then, given x majorized by y we have a recursive algorithm to construct a symmetric matrix $H = H_{K-1}$ with vector of diagonal entries x and vector of eigenvalues y. We only need to prove our claim in (16). We shall prove (16) by induction. The statement is true for $i = 0$ by definition. Let (16) be true for some $0 \le i < K - 1$. Now consider the following claim, for each $0 \le i < K$

$$H_i(k_{i+1}, l_{i+1}) = 0. \tag{17}$$

If this is true, then it is easy to verify that

$$H_{i+1}(k_{i+1}, k_{i+1})$$
$$= \alpha_{i+1} H_i(k_{i+1}, k_{i+1}) + (1 - \alpha_{i+1}) H_i(l_{i+1}, l_{i+1})$$
$$= \alpha_{i+1} y_{k_{i+1}}^{(i)} + (1 - \alpha_{i+1}) y_{l_{i+1}}^{(i)} = y_{k_{i+1}}^{(i+1)}$$
$$H_{i+1}(l_{i+1}, l_{i+1})$$
$$= \alpha_{i+1} H_i(l_{i+1}, l_{i+1}) + (1 - \alpha_{i+1}) H_i(k_{i+1}, k_{i+1})$$
$$= \alpha_{i+1} y_{l_{i+1}}^{(i)} + (1 - \alpha_{i+1}) y_{k_{i+1}}^{(i)} = y_{l_{i+1}}^{(i+1)}.$$

Hence H_{i+1} has vector of diagonal entries $y^{(i+1)}$. This completes the proof of the claim in (16) by induction. We show

(17), by first showing that the following stronger statement is true for each $1 \le i < K - 1$:

$$H_i(m, n) \ne 0 \text{ for } m < n \rightarrow m, n \in \{k_1, \cdots, k_i, l_1, \cdots l_i\}. \quad (18)$$

An appeal to (15) coupled with the claim in (18) above then shows that (17) is true. We show (18) by induction. Expression (18) is easily verified to be true for $i = 1$: only $H_1(k_1, l_1)$ and $H_1(l_1, k_1)$ can be nonzero among nondiagonal entries of H_1. Suppose (18) is true for some $1 \le i < K - 1$. Now consider the following cases.

1) $m \ne k_{i+1}$ and $n \ne l_{i+1}$. Then $H_{i+1}(m, n) = H_i(m, n)$ and (18) is true for $i + 1$ by the induction hypothesis.
2) $m = k_{i+1}$ and $n \ne l_{i+1}$. In this case, observe that

$$H_{i+1}(k_{i+1}, n) = \sum_{m_1, m_2} U_{i+1}(m_1, k_{i+1}) H_i(m_1, m_2)$$
$$\cdot U_{i+1}(m_2, n) \quad (19)$$

Since $n > m = k_{i+1}$ and $n \le l_{i+1}$, we have

$$U_{i+1}(m_2, n) = 0, \qquad \text{if } m_2 \ne n.$$

Continuing from (19), since the (n, n) entry of U_{i+1} is unity

$$H_{i+1}(k_{i+1}, n) = \sum_{m_1} U_{i+1}(m_1, k_{i+1}) H_i(m_1, n)$$
$$= \sqrt{\alpha_{i+1}} H_i(k_{i+1}, n) + \sqrt{1 - \alpha_{i+1}}$$
$$\cdot H_i(l_{i+1}, n).$$

Now if $H_{i+1}(k_{i+1}, n) \ne 0$ then at least one of $H_i(k_{i+1}, n)$ and $H_i(l_{i+1}, n)$ is nonzero and (18) is true for $i + 1$ by the induction hypothesis.
3) $n = l_{i+1}$ and $m \ne k_{i+1}$. This situation is analogous to the one above and an identical argument can be used to show that (18) is true for $i + 1$.

This completes the proof of (18) and the construction procedure is validated. In the next subsection we utilize this general construction procedure to construct optimal signature sequences.

B. Construction of Signature Sequences

Let $\mathcal{K} = \Phi$, i.e., no user is oversized. Then the vector $y = (p_{\text{tot}}, p_{\text{tot}}, \cdots, p_{\text{tot}}, 0, \cdots, 0)$ with $K - N$ entries equal to 0, majorizes the vector $x = (Np_1, \cdots, Np_K)$. Let $y^{(0)} = y$ and $H_0 = \text{diag}\{y_1, \cdots y_K\}$. Following the algorithm in the preceding subsection we have the sequence of unitary matrices $U_1, \cdots, U_{K-1}$ such that the $K \times K$ symmetric matrix $H = U_{K-1}^t \cdots U_1^t H_0 U_1 \cdots U_{K-1}$ has diagonal entries $Np_1, Np_2, \cdots, Np_K$ and N eigenvalues (of multiplicity both algebraic and geometric) equal to p_{tot} and $K - N$ null eigenvalues. Let $U = U_1 U_2 \cdots U_{K-1}$ Then the first N rows of U (say, $v_1, \cdots v_N$) are the normalized eigenvectors of H corresponding to the eigenvalue p_{tot} and denote $V^t = \{v_1 \cdots v_N\}$. Then we can write $H = p_{\text{tot}} V^t V$. As before, let $D = \text{diag}\{p_1, \cdots, p_K\}$. Define the $N \times K$ matrix $S = \sqrt{p_{\text{tot}}} V D^{-\frac{1}{2}}$. Since the diagonal entries of $S^t S$ are all equal to N, we have $S \in \mathcal{S}$. Furthermore, N eigenvalues of $D^{\frac{1}{2}} S^t S D^{\frac{1}{2}} = H$ are p_{tot} and $K - N$ eigenvalues are null (notice that, by construction, $SDS^t = p_{\text{tot}} I$). Thus for this choice of signature sequences S, we have, from (10), that

$$\mathcal{C}_{\text{sum}} = \frac{1}{2} \log \left(1 + \frac{p_{\text{tot}}}{\sigma^2}\right).$$

The following example illustrates this construction procedure.

Example 4.1: Three users, processing gain 2, power constraints $p_1 \ge p_2 \ge p_3$ and $p_{\text{tot}} \ge 2p_1$.

Condition $p_{\text{tot}} \ge 2p_1$ is the same as the condition that the sum of any two power constraints is lower-bounded by the third power constraint. We let $y = y^{(0)} = (p_{\text{tot}}, p_{\text{tot}}, 0)^t$ and $x = (2p_1, 2p_2, 2p_3)^t$. Following the algorithmic procedure outlined earlier, we have $\lambda_1 = \frac{2p_1}{p_{\text{tot}}}$ and $k_1 = 1$, $l_1 = 3$. Hence $y^{(1)} = (2p_1, p_{\text{tot}}, p_{\text{tot}} - 2p_1)^t$. In the second stage, $\lambda_2 = \frac{p_{\text{tot}} - 2p_3}{2p_1}$ and $k_2 = 2$, $l_2 = 3$. Hence $y^{(2)} = x = (2p_1, 2p_2, 2p_3)^t$ and the unitary matrices

$$U_1 = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \sqrt{(1-\lambda_1)} \\ 0 & 1 & 0 \\ -\sqrt{(1-\lambda_1)} & 0 & \sqrt{\lambda_1} \end{bmatrix}$$

$$\text{and} \quad U_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & \sqrt{1-\lambda_2} \\ 0 & -\sqrt{(1-\lambda_2)} & \sqrt{\lambda_2} \end{bmatrix}.$$

Hence we have the matrices at the bottom of this page.
The signature sequence matrix S then is as shown in the second matrix at the bottom of this page.

$$U = U_1 U_2 = \begin{bmatrix} \sqrt{\lambda_1} & -\sqrt{(1-\lambda_1)(1-\lambda_2)} & \sqrt{\lambda_2(1-\lambda_1)} \\ 0 & \sqrt{\lambda_2} & \sqrt{1-\lambda_2} \\ -\sqrt{1-\lambda_2} & -\sqrt{\lambda_1(1-\lambda_2)} & \sqrt{\lambda_1 \lambda_2} \end{bmatrix}$$

$$\text{and} \quad V = \begin{bmatrix} \sqrt{\lambda_1} & -\sqrt{(1-\lambda_1)(1-\lambda_2)} & \sqrt{\lambda_2(1-\lambda_1)} \\ 0 & \sqrt{\lambda_2} & \sqrt{1-\lambda_2} \end{bmatrix}$$

$$S = \sqrt{p_{\text{tot}}} V D^{-(1/2)} = \begin{bmatrix} \sqrt{2} & -\sqrt{\dfrac{(p_{\text{tot}} - 2p_1)(p_{\text{tot}} - 2p_2)}{2p_1 p_2}} & \sqrt{\dfrac{(p_{\text{tot}} - 2p_1)(p_{\text{tot}} - 2p_3)}{2p_1 p_3}} \\ 0 & \sqrt{\dfrac{p_{\text{tot}}(p_{\text{tot}} - 2p_3)}{2p_1 p_2}} & \sqrt{\dfrac{p_{\text{tot}}(p_{\text{tot}} - 2p_2)}{2p_1 p_3}} \end{bmatrix}$$

It is easily verified that the three signature sequences (the three columns of S) have norm $\sqrt{2}$ and hence $S \in \mathcal{S}$. It can also be verified that with this choice of S we have $SDS^t = p_{\text{tot}}I$. $\diamond$

In the special case when all the power constraints are equal (to say p), then $p_{\text{tot}} = Kp \geq Np$ and hence

$$\mathcal{C}_{\text{sum}} = \frac{1}{2} \log \left(1 + \frac{Kp}{\sigma^2} \right).$$

Furthermore, the sequence matrix S constructed above from H now satisfies the relation $SS^t = KI$. This result was observed in [7] and the sequences that meet this constraint were denoted WBE sequences (such sequences were also identified in [5]). Our algorithm specialized to this situation constructs WBE sequences for arbitrary $K \geq N$. Generalized WBE sequences also turn out to be optimal in a completely different context of maximizing the signal-to-interference ratio of the users in a single-cell power-controlled synchronous CDMA system with linear multiuser receivers in [10].

Now we focus on the general situation when $p_{\text{tot}} \geq Np_i$ for each i need not be satisfied. Without loss of generality, let $p_1 \geq p_2 \geq \cdots \geq p_K$. Let $e_i \in \mathbb{R}^N$ be the vector $(0, 0, \cdots, 0, 1, 0, \cdots, 0)$ with the entry 1 being in the ith position. Then $e_1, \cdots, e_N$ form an orthonormal basis for $\mathbb{R}^N$. Suppose the power constraints satisfy

$$\sum_{j=1}^{k-1} p_j + (N-k+1)p_k > p_{\text{tot}} \geq \sum_{j=1}^{k} p_j + (N-k)p_{k+1} \tag{20}$$

for some $k \in \{1, 2, \cdots, N-1\}$. Observe that (20) is always true for some unique (depending on the power constraints) k if $p_{\text{tot}} < Np_1$. A comparison with (6) shows that users $1, \cdots, k$ are oversized. Then, for $i = 1, \cdots, k$ let $s_i = \sqrt{N}e_i$. We shall now choose the signature sequences $s_{k+1}, \cdots, s_K$ for the remaining $K - k$ users from the subspace spanned by $\{e_{k+1}, \cdots, e_N\}$ which has dimension $N - k$. Since

$$\sum_{i=k+1}^{K} p_i \geq (N-k)p_{k+1}$$

from (20), we can appeal to the algorithm used previously to construct sequences $s_{k+1}, \cdots, s_K$ that have the property

$$\sum_{i=k+1}^{K} p_i s_i s_i^t = \left(\sum_{i=k+1}^{K} p_i \right) \begin{bmatrix} 0 & 0 \\ 0 & I_{N-k} \end{bmatrix}$$

where I_{N-k} is the identity matrix of dimension $(N-k) \times (N-k)$. It is easily verified that with this choice of signature sequences the sum capacity (given in (7))

$$\mathcal{C}_{\text{sum}} = \frac{N-k}{2N} \log \left(1 + \frac{N \left(\sum_{i=k+1}^{K} p_i \right)}{(N-k)\sigma^2} \right) + \frac{1}{2N} \sum_{i=1}^{k} \log \left(1 + \frac{Np_i}{\sigma^2} \right)$$

is achieved. We illustrate this construction with a simple example.

Example 4.2: Three users, processing gain 2, power constraints $p_1 \geq p_2 \geq p_3$ and $p_1 > p_2 + p_3$.

With these values of the power constraints, by an appeal to Theorem 3.1 we have

$$\mathcal{C}_{\text{sum}} = \frac{1}{4} \log \left(1 + \frac{2p_1}{\sigma^2} \right) + \frac{1}{4} \log \left(1 + \frac{2(p_2 + p_3)}{\sigma^2} \right). \tag{21}$$

Let e_1, e_2 be an orthonormal basis in $\mathbb{R}^2$. Following the algorithmic procedure above we let $s_1 = \sqrt{2}e_1$ and $s_2 = s_3 = \sqrt{2}e_2$. With this choice of signature sequences, i.e., when $S = \{\sqrt{2}e_1 \sqrt{2}e_2 \sqrt{2}e_2\}$, it is trivially verified that the maximum sum rate point in the capacity region $\mathcal{C}(S)$ equals $\mathcal{C}_{\text{sum}}$ in (21). $\diamond$

V. SUMMARY AND CONCLUSION

We have completely characterized the sum capacity of a multiuser synchronous CDMA system. This characterization allowed us to derive necessary and sufficient conditions on the power constraints of the users so that for some choice of signature sequences the sum capacity of the system equals that of the system with no spreading, namely,

$$C = \frac{1}{2} \log \left(1 + \frac{p_{\text{tot}}}{\sigma^2} \right).$$

We also identified the signature sequences that achieved sum capacity and proposed a simple algorithm to construct them. A byproduct of the construction scheme is the following simple summarizing interpretation: Let the power constraints of the K users satisfy $p_1 \geq p_2 \geq \cdots \geq p_K$.

1) Step 1: If $K \leq N$ the sum capacity is

$$\mathcal{C}_{\text{sum}} = \frac{1}{2N} \sum_{i=1}^{K} \log \left(1 + \frac{Np_i}{\sigma^2} \right).$$

The choice of orthogonal signature sequences for the users is optimal and achieves sum capacity. Unless $K = N$ and $p_1 = p_2 = \cdots = p_K = p$, the sum capacity is strictly less than

$$C = \frac{1}{2} \log \left(1 + \frac{p_{\text{tot}}}{\sigma^2} \right).$$

2) Step 2: Let $K > N$ henceforth. If $p_{\text{tot}} \geq Np_1$, then

$$\mathcal{C}_{\text{sum}} = C = \frac{1}{2} \log \left(1 + \frac{p_{\text{tot}}}{\sigma^2} \right).$$

The algorithm we derived in Section IV can be used to construct signature sequences which achieve sum capacity.

3) Step 3: Suppose $Np_1 > p_{\text{tot}}$. Then we let user 1 have an independent channel (we do this by letting s_1 orthogonal to all the other signature sequences) and then reduce the problem to $K - 1$ users in a system with processing gain $N - 1$. The resulting sum capacity is strictly smaller than C.

The allocation of signature sequences so as to achieve sum capacity emphasizes the "unfairness" of the performance criterion $\mathcal{C}_{\mathrm{sum}}$. When

$$\sum_{i=1}^{N-2} p_i + 2p_{N-1} > p_{\mathrm{tot}}$$

the first $N-1$ users (these users have the largest values of p_i and hence the weakest power constraints) are allocated orthogonal signature sequences (independent channels and are hence allowed to transmit at higher rates) while the other $K-N+1$ users share a common channel. In a wireless system, some users might be far away from the base station and their received powers will be correspondingly lower than users which are close to the base station. It is desirable in a practical system that resource allocation based on some choice of performance criterion of the system be "fair" to the users. Hence it is important to address fairly the situation of asymmetric power constraints on the users. One way to do this is to change the performance criterion to a weighted linear combination of the rates of the users, the weights in inverse proportion to the path gains of the users to the base station. A second way is to consider the "symmetric capacity" defined in [7]. The symmetric capacity is the sum rate of the maximum achievable equal-rate point in the union capacity region $\mathcal{C} = \cup_{S \in \mathcal{S}} \mathcal{C}(S)$. These questions will be answered if the characterization of the entire union capacity region $\mathcal{C}$ is done. Our current efforts are directed towards solving this important open question.

In this paper we have focussed on symbol-synchronous CDMA systems. Indeed, most existing capacity results except [1], [9] pertain to the symbol-synchronous case. The extension of our results to the asynchronous situation is also interesting and is an important open problem.

APPENDIX
PROOF OF LEMMA 4.1

We reproduce here for completeness the proof that if x is majorized by y then x may be derived from y by successive applications of T-transforms (utmost $K-1$ applications) from the classical text [3].

Let x be majorized by y. We assume that x is not obtainable from y by permuting elements of y, else the statement is trivially true. Without loss of generality, let $x_1 \geq \cdots \geq x_K$ and $y_1 \geq \cdots \geq y_K$. Let j be the largest index such that $x_j < y_j$, and let k be the smallest index greater than j such that $x_k > y_k$. Such a pair j, k must exist, since the largest index i for which $x_i \neq y_i$ must satisfy $x_i > y_i$. By choice of j and k, we have $y_j > x_j \geq x_k > y_k$. Let $\delta = \min(y_j - x_j, x_k - y_k)$ and $1 - \alpha = \frac{\delta}{y_j - y_k}$ and let

$$y^* = (y_1, \cdots, y_{j-1}, y_j - \delta, y_{j+1}, \cdots, y_{k-1}, y_k + \delta, y_{k+1}, \cdots, y_K).$$

It is easy to verify that $\alpha \in (0,1)$ and that

$$y^* = \alpha y + (1-\alpha)(y_1, \cdots, y_{j-1}, y_k, y_{j+1}, \cdots, y_{k-1}, y_j, y_{k+1}, \cdots, y_K).$$

Thus $y^* = Ty$ for $T = \alpha I + (1-\alpha)Q$, where Q interchanges the jth and kth coordinates. The claim is that y^* majorizes x. To see this, note that

$$\sum_{s=1}^{l} y_s^* = \sum_{s=1}^{l} y_s \geq \sum_{s=1}^{l} x_s, \qquad l = 1, \cdots, j-1$$
$$y_j^* \geq x_j, y_s^* = y_s, \qquad s = j+1, \cdots, k-1$$
$$\sum_{s=1}^{l} y_s^* = \sum_{s=1}^{l} y_s \geq \sum_{s=1}^{l} x_s, \qquad l = k+1, \cdots, K$$
$$\sum_{s=1}^{K} y_s^* = \sum_{s=1}^{K} y_s = \sum_{s=1}^{K} x_s.$$

For any two vectors u, v let $d(u,v)$ be the number of nonzero differences $u_i - v_i$. Since $y_j^* = x_j$ if $\delta = y_j - x_j$ and $y_k^* = x_k$ if $\delta = x_k - y_k$, it follows that $d(x, y^*) \leq d(x,y) - 1$. Hence, x can be derived from y by successive applications of a finite number of T-transformations. Since $d(x,y) \leq K$ and $d(x,y) \neq 1$ (otherwise, $\sum_{s=1}^{K} x_s \neq \sum_{s=1}^{K} y_s$) at most $(K-1)$ T-transformations are required. $\square$

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Bounds on the Information Rate of Intertransition-Time-Restricted Binary Signaling Over an AWGN Channel

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Abstract— Upper and lower bounds on the capacity of a continuous-time additive white Gaussian noise (AWGN) channel with bilevel $(\pm\sqrt{P})$ input signals subjected to a minimum inter-transition time $(T_{\min})$ constraint are derived. The channel model and input constraints reflect basic features of certain magnetic recording systems. The upper bounds are based on Duncan's relation between the average mutual information in an AWGN regime and the mean-square error (MSE) of an optimal causal estimator. Evaluation or upper-bounding the MSE of suboptimal causal estimators yields the desired upper bounds. The lower bound is found by invoking the extended "Mrs. Gerber's" Lemma and asymptotic properties of the entropy of max-entropic bipolar (d, k) codes.

Asymptotic results indicate that at low SNR = $PT_{\min}/N_0$, with N_0 designating the noise one-sided power spectral density, the capacity tends to P/N_0 nats per second (nats/s), i.e., it equals the capacity in the simplest average power limited case. At high SNR, the capacity behaves asymptotically as $T_{\min}^{-1}\ln(\text{SNR}/\ln(\text{SNR}))$ (nats/s), demonstrating the degradation relatively to $T_{\text{avg}}^{-1}\ln\text{SNR}$, which is the asymptotic known behavior of the capacity with a bilevel average intertransition-time (T_{avg}) restricted channel input. Additional lower bounds are obtained by considering specific signaling formats such as pulsewidth modulation. The effect of mild channel filtering on the lower bounds on capacity is also addressed, and novel techniques to lower-bound the capacity in this case are introduced.

Index Terms— Channel capacity, constrained bipolar inputs, information rates, intertransition time, magnetic recording, Mrs. Gerber's lemma.

I. INTRODUCTION

THE capacity of a filtered additive white Gaussian noise (AWGN) channel with bipolar (bilevel binary) inputs, depicted in Fig. 1 where $s(t)$, $r(t)$, and $n(t)$ stand, respectively, for the input, output, and the AWGN processes, and where $g(t)$ is the channel filter impulse response, has been intensively studied with increased interest in the last decade. This input constraint characterizes a variety of communication systems, and in particular is relevant to most variants of magnetic and optical recording systems [1]. The dominant motivation for this endeavor is to try and capture the information-theoretic implications of the bipolar input constraint and thus provide
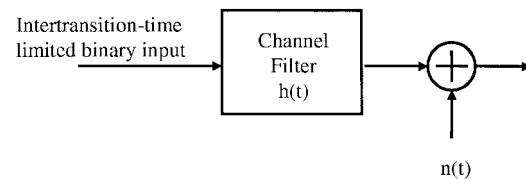
Fig. 1. Channel model.

improved more realistic predictions of the ultimate performance limits of the relevant information-conveying systems.

It is known that with no channel filter (that is, an infinite bandwidth channel) employing bipolar input (taking on the values $\pm\sqrt{P}$) does not reduce the classical average-power limited, AWGN infinite bandwidth channel capacity P/N_0 nats per second (nats/s) where P stands for the input power and N_0 stands for the power spectral density of the AWGN [2, and references therein]. In [3] it has been proved that for any square integrable channel filter impulse response the bipolar constrained input attains the same capacity as is achieved by an arbitrary peak limited inputs, that is, $|x(t)| \le \sqrt{P}, \forall t$. Lower bounds on the capacity were derived in [3] and tightened in [4] and [5]. Upper bounds, which are strictly lower than the average-power constrained capacity were presented in [6]. The channel introduced in [3] (see Fig. 1) has been suggested as a simplified model of a certain magnetic recording systems and capacity calculations for specific parameters were reported in [7] and [8].

In [3] and [6] no further constraints were imposed on the bipolar inputs and the very basic result in [3] on the equivalence of the capacities of peak-power-limited and bipolar inputs, implies an unbounded transition rate bipolar input. This physically impractical demand fails to capture the "bandwidth-like" limitation of the input process imposed by practical considerations and inherent system restrictions. In [9] the effect of the average transition rate of the bipolar input process on upper bounds on capacity has been addressed. The random telegraph bipolar input with a given transition rate has been considered in [2] where its asymptotic (signal to noise ratio (SNR) $\to \infty$) optimality is established under an average (rather than minimal) intertransition duration constraint.

A natural constraint on the temporal variation of a bipolar input is the minimal duration between transitions, that is, the time between consecutive transitions of the input signals is no shorter than $T_{\min}$ seconds. This is a typical constraint in a magnetic recording system which is aimed to prohibit

closely spaced transitions as to mitigate heavy deleterious intersymbol interference (ISI) effects [1]. Further, the linear channel model for magnetic recording collapses when closely spaced consecutive transitions take place [3], [10] and nonlinear intersymbol interference emerges. The well-known (d, ∞) (or runlength-limited) codes [1], [11] constitute the discrete-time version of the minimum intertransition time-constrained signals and the capacity of these has been examined for the discrete-time AWGN channel in [12] and [13]. In [14] and [15], runlength-limited codes were examined in a channel, the output of which comprises jittered (noisy) observations of the transition instants. A guard-space random telegraph process which satisfies the minimal intertransition duration (MID) constraint $(T_{\min})$ has been considered in [2], where asymptotic (high-SNR) results for the capacity were reported. Related results on the capacity and cutoff rates of filtered continuous-time and discrete-time AWGN channels with peak-power-limited and/or slope-limited input signals can be found in [16]–[18] and references therein.

In this work we focus on bipolar inputs with the MID $(T_{\min})$ constraint and investigate the effect of $T_{\min}$ on C_{MID}, the achievable information capacity of the AWGN channel. We mainly specialize to the infinite bandwidth channel (that is, no channel filter) leaving thus $T_{\min}$ to reflect the basic temporal-variation restriction of the input waveform. In the next section, the asymptotic behavior of the MID-constrained capacity is addressed in view of the results in [2]. In Section III, upper bounds on the capacity are evaluated. The bounds are based on Duncan's theorem [19] which relates in the AWGN regime the average mutual information to the mean-square error (MSE) of the optimal causal estimator. Linear, suboptimum nonlinear, and improved nonlinear estimators are introduced, and the resultant upper bounds on capacity are explicitly derived. Lower bounds on capacity are found in Section IV, where results on maxentropic runlength-limited sequences are combined with the extended "Mrs. Gerber's" Lemma [20]. Comparisons are made to the achievable information rates of binary pulse amplitude modulation (PAM) and pulsewidth-modulation (PWM) signaling as well as to the random telegraph wave with and without a guard time interval.

Among other results it is concluded that for asymptotically high values of $\mathrm{SNR} = PT_{\min}/N_0$ ($\mathrm{SNR} \to \infty$) the capacity behaves like $T_{\min}^{-1} \ln(\mathrm{SNR}/\ln \mathrm{SNR})$ nats/s, while for asymptotically low SNR values ($\mathrm{SNR} \to 0$) the expected P/N_0 (nats/s) behavior is evidenced. In Section IV, the effect of a mild lowpass channel filter on the capacity lower bounds is considered. This result is found by embedding the basic bounding technique employed in Section IV which is based on the extended "Mrs. Gerber's" Lemma into the Shamai–Ozarow–Wyner (SOW) lower bound [4] on the capacity of a discrete time AWGN channel with ISI. For a mild window-integrator channel filter of integration time $T_F < T_{\min}$, lower bounds on capacity are found using the Fano inequality along with upper bounds on the error probability of a set of carefully selected equi-energy signals. The later

bounds exhibit an asymptotic ($\mathrm{SNR} \to \infty$) behavior of

$$\frac{1}{2}T_{\min}^{-1} \ln \left(\frac{\mathrm{SNR}}{\ln \mathrm{SNR}} \left(\frac{4e}{\pi} \right) \left(\frac{T_{\min}}{T_F} \right) \right)$$

where the factor $1/2$ is attributed to the smoothing effect of the channel filter [2] and $T_{\min}/T_F$ represents the normalized filtering bandwidth.

II. HIGH-SNR ASYMPTOTICS

To the end of comparing various bounds on MID-constrained capacity to be derived in this paper, we present first the asymptotic high-SNR behavior of this capacity, relying on the results and methods discussed in [2].

In [2], the case of T_{avg}-limited bipolar signaling was addressed. High-SNR treatment revealed that the asymptotically capacity-achieving waveform is a random telegraph waveform (RTW). The capacity was shown to behave as $T_{\mathrm{avg}}^{-1} \ln(kT_{\mathrm{avg}}/\sigma_t)$ where the σ_t is the RMS error for estimating the time of transition, and the factor k is related to the distribution of this error. In the case of unfiltered channel, the distribution is a two-sided exponential, $\sigma_t = N_0/2P = T_{\mathrm{avg}}/2\mathrm{SNR}$ and $k = 1/2$. In the case of filtered channel, the transition instant estimation error is Gaussian, and hence $\sigma_t = \sqrt{N_0/2\beta^2 P} \propto T_{\mathrm{avg}}/\sqrt{\mathrm{SNR}}$ and $k = \sqrt{e/2\pi}$, where

$$\beta^2 = (1/P) \int_{-\infty}^{\infty} g(t)g''(t)\,dt$$

and $g(t)$ is the transition shape. In the case of linear slope transitions (rectangular impulse response filter) of duration T_F, an example to be used later yields $\beta^2 = 4/T_F$ and

$$\sigma_t = \sqrt{T_F N_0/8P} = \sqrt{T_{\mathrm{avg}}T_F/8\mathrm{SNR}}.$$

Further, [2] addressed the $T_{\min}$-limited case and showed the high-SNR behavior to be $T_{\mathrm{avg}}^{-1} \ln(k(T_{\mathrm{avg}} - T_{\min})/\sigma_t)$, with the signaling waveform being the guard-time RTW.

Capacity is obtained by maximizing the information transfer ($\mathcal{IT}$) rate with respect to signal distribution. Assuming that for the given $T_{\min}$ and T_{avg}, $\mathcal{IT}$ is maximized by guard-time RTW, and is, as mentioned, given by $T_{\mathrm{avg}}^{-1} \ln(k(T_{\mathrm{avg}} - T_{\min})/\sigma_t)$, the $\mathcal{IT}$ for the $T_{\min}$-limited case is obtained by maximization of $\mathcal{IT}(T_{\min}, T_{\mathrm{avg}}, \mathrm{SNR})$ with respect to $T_{\mathrm{avg}} \geq T_{\min}$. Let us define a function $F(\alpha)$ by

$$F(\alpha) = \max_{x > 1} x^{-1} \ln \alpha(x - 1). \tag{1}$$

It follows then that

$$\mathcal{IT}(T_{\min}, \mathrm{SNR}) = T_{\min}^{-1} F(kT_{\min}/\sigma_t). \tag{2}$$

The asymptotic behavior of $F(\alpha)$ is studied in Appendix B and is shown to be $F(\alpha) \approx \ln(\alpha/\ln \alpha)$. From here we can immediately deduce that the high-SNR behavior of the MID capacity in the unfiltered case is $\mathcal{IT}(T_{\min}, \mathrm{SNR}) \approx T_{\min}^{-1} \ln(\mathrm{SNR}/\ln \mathrm{SNR})$, and in the mildly filtered case the behavior is

$$\mathcal{IT}(T_{\min}, \mathrm{SNR}) \approx T_{\min}^{-1}(1/2) \ln(\mathrm{SNR}/\ln \mathrm{SNR}).$$

In the following these asymptotics will be used as a baseline to which the various bounds will be compared.